

# THREE LECTURES ON LOCAL COHOMOLOGY MODULES SUPPORTED ON MONOMIAL IDEALS

JOSEP ÀLVAREZ MONTANER

ABSTRACT. These notes are an extended version of a set of lectures given at "MONICA: MONomial Ideals, Computations and Applications", at the CIEM, Castro Urdiales (Cantabria, Spain) in July 2011. The goal of these lectures is to give an overview of some results that have been developed in recent years about the structure of local cohomology modules supported on a monomial ideal. We will highlight the interplay of multi-graded commutative algebra, combinatorics and D-modules theory that allow us to give different points of view to this subject. We will try to preserve the informal character of the lectures, so very few complete proofs are included.

## CONTENTS

<b>Introduction</b>	2
<b>Lecture 1: Basics on local cohomology modules</b>	4
1. Basic definitions	4
2. The theory of $D$ -modules	8
<b>Lecture 2: Local cohomology modules supported on monomial ideals</b>	17
3. $\mathbb{Z}^n$ -graded structure	18
4. $D$ -module structure	22
5. Building a dictionary	40
<b>Lecture 3: Bass numbers of local cohomology modules</b>	47
6. Bass numbers of modules with variation zero	48
7. Injective resolution of $H_I^r(R)$	54
<b>Appendix: <math>\mathbb{Z}^n</math>-graded free and injective resolutions</b>	56

<b>Tutorial (joint with O. Fernández-Ramos)</b>	60
8. Getting started	62
9. Tutorial	65
References	67

## Introduction

Local cohomology was introduced by A. Grothendieck in the early 1960's and quickly became an indispensable tool in Commutative Algebra. Despite the effort of many authors in the study of these modules, their structure is still quite unknown. C. Huneke [44] raised some basic questions concerning local cohomology:

- Annihilation of local cohomology modules.
- Finitely generation of local cohomology modules.
- Artinianity of local cohomology modules.
- Finiteness of the associated primes set of local cohomology modules.

In general, one can not even say when they vanish. Moreover, when they do not vanish they are rarely finitely generated. However, in some situations these modules verify some finiteness properties that provide a better understanding of their structure.

For example, when we have a regular local ring containing a field, then the set of associated primes of local cohomology modules is finite. Moreover, the Bass numbers of these modules with respect to any prime ideal are also finite. This result has been proved by C. Huneke and R. Y. Sharp [45] (see also [57]) in positive characteristic and by G. Lyubeznik in the zero characteristic case [56].

The key idea behind these results is that we can enrich our local cohomology modules with an extra structure that makes easier to describe them (see [58] for a nice survey). In characteristic zero we have that local cohomology modules are finitely generated modules over the ring of differential operators, in fact they are holonomic. In positive characteristic we have that local cohomology modules are  $F$ -finite  $F$ -modules. This is a notion developed by G. Lyubeznik in [57] with the help of the Frobenius morphism. It is proved in [57] that this class of modules satisfy analogous properties to those of holonomic  $D$ -modules.

In this survey we turn our attention to the case of local cohomology modules supported on monomial ideals. The interest for this kind of modules is not new since it provides a nice family of examples that are easier to deal with when one considers some of the basic

problems concerning local cohomology but also because their graded pieces can be used to describe cohomology groups of sheaves on a toric variety (see [67],[24]).

A lot of progress in the study of these modules has been made in recent years based on the fact that they have a structure as  $\mathbb{Z}^n$ -graded module. We will give an overview of this approach but we will put more emphasis on the  $D$ -module approach because of the habit of the author but also because it gives a new perspective for the reader that has a stronger background in Combinatorial Commutative Algebra.

In the first part of the first Lecture we will review the basics on the theory of local cohomology modules we will use throughout this work. The second part of the first Lecture is devoted to the theory of  $D$ -modules. We start from scratch with the basic definitions and we illustrate them with the main examples we will consider in this work: the polynomial ring, localizations of this polynomial ring and local cohomology modules. Then we quickly direct ourselves to the construction of the *characteristic cycle*. This is an invariant that gives us a lot of information about our module. Again, we will illustrate this fact with some basic examples.

In the second Lecture we will study the structure of local cohomology modules with support a monomial ideal from two different point of views. First we will highlight the main results obtained in [66], [81], [87] where the  $\mathbb{Z}^n$ -graded structure of these modules is described. Then we will consider the  $D$ -module approach given in [1], [5], [6]. We tried to illustrate this approach with a lot of examples since we assume that the reader is not familiar with this point of view. In the end, it turns out that both approaches are equivalent and one can build a dictionary between the  $\mathbb{Z}^n$ -graded and the  $D$ -module structure of these local cohomology modules. More striking is the dictionary we will build between local cohomology modules and free resolutions and we will extend in the third Lecture.

In the third Lecture we turn our attention to Bass numbers of local cohomology modules with support a monomial ideal. A formula to describe these invariants using the  $\mathbb{Z}^n$ -graded structure has been given by K. Yanagawa in [87]. From the  $D$ -modules point of view we have to refer to the algorithms given by the author in [1], [3]. However, in this survey we will mainly report some recent work by A. Vahidi and the author [8]. The approach given in [8] is to consider the pieces of the composition of local cohomology modules. It turns out that the structure of these local cohomology modules (as  $\mathbb{Z}^n$ -graded module or module with variation zero) is required to compute their Bass numbers. Using the dictionary between local cohomology modules and free resolutions, one may understand the Lyubeznik numbers as a measure for the acyclicity of the linear strands of the free resolution of the Alexander dual ideal.

Bass numbers of local cohomology modules do not behave as nicely as in the case of finitely generated modules but we can control them using their structure of the local cohomology modules. This control leads to some partial description of its injective resolution. In particular we give a bound for the injective dimension in terms of the small support of these modules.

We also included an Appendix where we fix the notations and some basic facts on  $\mathbb{Z}^n$ -graded free and injective resolutions. We recommend the reader to take a look at this appendix whenever one finds some unexplained reference to these concepts.

In the final Section we provide some functions developed with O. Fernández Ramos using `Macaulay 2` that allows us to compute the characteristic cycle of local cohomology modules supported on monomial ideals and also Lyubeznik numbers. We also included a Tutorial aimed to start experimenting with an small sample of exercises. Once this set of exercises is completed we encourage the reader to develop with their own examples and come out with their own results.

## Lecture 1: Basics on local cohomology modules

### 1. BASIC DEFINITIONS

We will start with a quick introduction to the theory of local cohomology modules as introduced in [38]. Our main reference will be the book of M.P. Brodmann and R.Y. Sharp [17] but we also recommend to take a look at [22] and [46]. Then we will turn our attention to some finiteness results on these modules that prompted G. Lyubeznik [56] to define a new set of invariants that we will study in Lecture 3.

Let  $I \subseteq R$  be an ideal of a commutative Noetherian ring  $R$ . The usual way to introduce local cohomology modules is through *the  $I$ -torsion functor* over the category of  $R$ -modules. This is a functor  $\Gamma_I : \text{Mod}(R) \rightarrow \text{Mod}(R)$  defined for any  $R$ -module  $M$  as:

$$\Gamma_I(M) := \{x \in M \mid I^n x = 0 \text{ for some } n \geq 1\}.$$

The category  $\text{Mod}(R)$  of  $R$ -modules is abelian and has enough injectives. On the other hand, the  $I$ -torsion functor  $\Gamma_I$  is additive, covariant and left-exact so it makes sense to consider the right derived functors of  $\Gamma_I$ . These are called *the local cohomology modules of  $M$  with respect to  $I$*  and are denoted by

$$H_I^r(M) := \mathbb{R}^r \Gamma_I(M).$$

**Remark 1.1.** *The functor of  $I$ -torsion can be expressed as*

$$\Gamma_I(M) = \bigcup_{n \geq 0} (0 :_M I^n) = \varinjlim \text{Hom}_R(R/I^n, M),$$

so we have  $H_I^r(M) = \varinjlim \operatorname{Ext}_R^r(R/I^n, M)$ .

Throughout this work, we will prefer to describe these local cohomology modules by means of the *Čech complex*. It will help us to put our hands on these objects and make some explicit computations but it will be also helpful when we want to enrich the local cohomology modules with an extra structure as a  $D$ -module structure, a Frobenius action structure or, in the case of monomial ideals, a  $\mathbb{Z}^n$ -graded structure.

Assume that our ideal  $I \subseteq R$  is generated by  $f_1, \dots, f_s$ . For any  $R$ -module  $M$ , the Čech complex of  $M$  with respect to  $I$ , that we will denote  $\check{C}_I^\bullet(M)$  is the following complex:

$$0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{1 \leq i_1 \leq s} M_{f_{i_1}} \xrightarrow{d_1} \dots \xrightarrow{d_{p-1}} \bigoplus_{1 \leq i_1 < \dots < i_p \leq s} M_{f_{i_1} \dots f_{i_p}} \xrightarrow{d_p} \dots \xrightarrow{d_{s-1}} M_{f_1 \dots f_s} \longrightarrow 0,$$

where the differentials  $d_p$  are defined by using the canonical localization morphism on every component  $M_{f_{i_1} \dots f_{i_p}} \longrightarrow M_{f_{j_1} \dots f_{j_{p+1}}}$  as follows:

$$d_p(m) = \begin{cases} (-1)^k \frac{m}{1} & \text{if } \{i_1, \dots, i_p\} = \{j_1, \dots, \widehat{j_k}, \dots, j_{p+1}\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the local cohomology modules are nothing but the cohomology modules of this complex, i.e.

$$H_I^r(M) = H^r(\check{C}_I^\bullet(M)) = \ker d_r / \operatorname{Im} d_{r-1}$$

The following basic properties will be often used without further mention. The main one states that local cohomology only depends on the radical of the ideal.

- $H_I^r(M) = H_{\operatorname{rad}(I)}^r(M)$ , for all  $r \geq 0$ .
- Let  $\{M_j\}_{j \in J}$  be an inductive system of  $R$ -modules. Then:

$$H_I^r(\varinjlim M_j) = \varinjlim H_I^r(M_j).$$

- *Invariance with respect to base ring:* Let  $R \longrightarrow S$  be a homomorphism of rings. Let  $I \subseteq R$  be an ideal and  $M$  an  $S$ -module. Then:

$$H_{IS}^r(M) \cong H_I^r(M).$$

- *Flat base change:* Let  $R \longrightarrow S$  be a flat homomorphism of rings. Let  $I \subseteq R$  be an ideal and  $M$  an  $R$ -module. Then:

$$H_{IS}^r(M \otimes_R S) \cong H_I^r(M) \otimes_R S.$$

Local cohomology modules are in general not finitely generated as  $R$ -modules, so they are difficult to deal with. In order to extract some properties of these modules the usual method is to use several exact sequences or spectral sequences involving these modules. Enumerated are some examples we will use in this work, for details we refer to [17] .

- **Long exact sequence of local cohomology:** Let  $I \subseteq R$  be an ideal and  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  an exact sequence of  $R$ -modules. Then we have the exact sequence:

$$\cdots \rightarrow H_I^r(M_1) \rightarrow H_I^r(M_2) \rightarrow H_I^r(M_3) \rightarrow H_I^{r+1}(M_1) \rightarrow \cdots$$

- **Mayer-Vietoris sequence:** Let  $I, J \subseteq R$  be ideals and  $M$  an  $R$ -module. Then we have the exact sequence:

$$\cdots \rightarrow H_{I+J}^r(M) \rightarrow H_I^r(M) \oplus H_J^r(M) \rightarrow H_{I \cap J}^r(M) \rightarrow H_{I+J}^{r+1}(M) \rightarrow \cdots$$

- **Brodmann's sequence:** Let  $I \subseteq R$  be an ideal and  $M$  an  $R$ -module. For any element  $f \in R$  we have the exact sequence:

$$\cdots \rightarrow H_{I+(f)}^r(M) \rightarrow H_I^r(M) \rightarrow H_I^r(M)_f \rightarrow H_{I+(f)}^{r+1}(M) \rightarrow \cdots$$

- **Grothendieck's spectral sequence:** Let  $I, J \subseteq R$  be ideals and  $M$  an  $R$ -module. Then we have the spectral sequence:

$$E_2^{p,q} = H_J^p(H_I^q(M)) \Rightarrow H_{I+J}^{p+q}(M).$$

Quite recently, a generalization of the Mayer-Vietoris sequence has been given in [5]. We give the presentation<sup>1</sup> introduced in [60] since it requires less notation.

- **Mayer-Vietoris spectral sequence:** Let  $I = I_1 \cap \cdots \cap I_n$  be a decomposition of an ideal  $I \subseteq R$  and  $M$  an  $R$ -module. Then we have the spectral sequence:

$$E_1^{-p,q} = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq n} H_{I_{i_1} + \cdots + I_{i_p}}^q(M) \Rightarrow H_I^{q-p}(M).$$

**1.1. Finiteness results.** Although the local cohomology modules are in general not finitely generated, under certain conditions they satisfy some finiteness properties that provide a better understanding of their structure. A turning point in the theory came at the beginning of the 1990's with the following remarkable result:

Let  $R$  be any regular ring containing a field of characteristic zero and  $I \subseteq R$  is an ideal. Then, the local cohomology modules  $H_I^i(R)$  satisfy the following properties:

- 1)  $H_m^j(H_I^i(R))$  is injective, where  $m$  is any maximal ideal of  $R$ .
- 2)  $\text{id}_R(H_I^i(R)) \leq \dim_R H_I^i(R)$ .
- 3) The set of the associated primes of  $H_I^i(R)$  is finite.
- 4) All the Bass numbers of  $H_I^i(R)$  are finite.

This result was proved by C. Huneke and R.Y. Sharp [45] for regular rings containing a field of positive characteristic using the Frobenius map. In his paper [56], G. Lyubeznik proved the case of regular rings containing a field of characteristic zero using the theory

<sup>1</sup>The presentation given in [60] slightly differs from the one given in [5] at the  $E_1$ -page but they coincide at the  $E_2$ -page.

of algebraic  $D$ -modules. Even though local cohomology modules have been already used in the theory of  $D$ -modules, the work of G. Lyubeznik became the first application of this theory to an explicit problem in Commutative Algebra. The main point is that local cohomology modules  $H_I^i(R)$  are finitely generated as  $D$ -modules. In fact they are *holonomic*  $D$ -modules. This class of modules form an abelian subcategory of the category of  $D$ -modules with some good properties, in particular they have finite length.

The results of G. Lyubeznik are slightly more general than those of C. Huneke and R.Y. Sharp. To prove the validity of this generalization in positive characteristic he introduced the theory of  $F$ -modules in [57]. In particular he introduced the class of  $F$ -finite  $F$ -modules that satisfy analogous properties to those of holonomic  $D$ -modules. Local cohomology modules belong to this class so one can follow a similar program to prove the same results in positive characteristic.

**1.2. Lyubeznik numbers.** Using the finiteness of Bass numbers G. Lyubeznik [56] defined a new set of numerical invariants. More precisely, let  $(R, \mathfrak{m}, k)$  be a regular local ring of dimension  $n$  containing a field  $k$  and  $A$  a local ring which admits a surjective ring homomorphism  $\pi : R \rightarrow A$ . Set  $I = \text{Ker } \pi$ , then we consider the Bass numbers

$$\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H_I^{n-i}(R)).$$

This invariants depend only on  $A$ ,  $i$  and  $p$ , but neither on  $R$  nor on  $\pi$ . Completion does not change  $\lambda_{p,i}(A)$  so one can assume  $R = k[[x_1, \dots, x_n]]$ . It is worthwhile to point out that, by [56, Lemma 1.4], we have

$$\lambda_{p,i}(R/I) = \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^p(H_I^{n-i}(R))),$$

i.e.  $H_{\mathfrak{m}}^p(H_I^{n-i}(R)) \cong E(R/\mathfrak{m})^{\lambda_{p,i}}$ . This is the approach that we will use in Lecture 3.

Lyubeznik numbers satisfy  $\lambda_{d,d}(A) \neq 0$  and  $\lambda_{p,i}(A) = 0$  for  $i > d$ ,  $p > i$ , where  $d = \dim A$ . Therefore we can collect them in what we refer as *Lyubeznik table*:

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

They have some interesting topological interpretation, as it was already pointed out in [56] but not so many examples can be found in the literature. The basic ones being the following:

**Example:** Assume that  $H_I^r(R) = 0$  for all  $r \neq \text{ht } I$ . Then, using Grothendieck's spectral sequence to compute the composition  $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$ , we obtain a trivial Lyubeznik table.

$$\Lambda(R/I) = \begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}$$

This situation is achieved in the following cases:

- $R/I$  is a complete intersection.
- $R/I$  is Cohen-Macaulay and contains a field of positive characteristic.
- $R/I$  is Cohen-Macaulay and  $I$  is a squarefree monomial ideal.

**Example:** Assume that  $\text{Supp}_R H_I^r(R) \subseteq V(\mathfrak{m})$  for all  $r \neq \text{ht } I$ . Rings satisfying this property can be viewed as rings which behave cohomologically like an isolated singularity. Then, the corresponding Lyubeznik table has the following shape:

$$\Lambda(R/I) = \begin{pmatrix} 0 & \lambda_{0,1} & \lambda_{0,2} & \cdots & \lambda_{0,d-1} & 0 \\ & 0 & 0 & \cdots & 0 & 0 \\ & & 0 & \cdots & 0 & \lambda_{0,d-1} \\ & & & \ddots & \vdots & \vdots \\ & & & & 0 & \lambda_{0,2} \\ & & & & & \lambda_{0,1} + 1 \end{pmatrix}$$

That is  $\lambda_{a,d} - \delta_{a,d} = \lambda_{0,d-a+1}$  for  $2 \leq a \leq d$ , where  $\delta_{a,d}$  is the Kronecker delta function. This result was proved in [32] for isolated singularities over  $\mathbb{C}$ . It was then generalized in any characteristic in [14], [13] (see also [77]).

## 2. THE THEORY OF $D$ -MODULES

In this section we will provide some basic foundations in the theory of modules over the ring of differential operators that we will use throughout this work. We will only scratch the surface of this theory so we encourage the interested reader to take a deeper look at the available literature. The main references that we will use are [12], [16] and [19].

Let  $k$  be a subring of a commutative Noetherian ring  $R$ . The ring of differential operators  $D_R$ , introduced by A. Grothendieck in [39, §16.8], is the subring of  $\text{End}_k(R)$  generated by the  $k$ -linear derivations and the multiplications by elements of  $R$ . In these lectures we will mainly consider the case where  $R$  is a polynomial over a field  $k$  of characteristic zero.

**2.1. The Weyl algebra.** Let  $R = k[x_1, \dots, x_n]$  be a polynomial ring over a field  $k$  of characteristic zero. The ring of differential operators  $D_R$  coincides with the Weyl algebra  $A_n(k) := k[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle$ , i.e. the non commutative  $R$ -algebra generated by the partial derivatives  $\partial_i = \frac{d}{dx_i}$ , with the relations given by:

$$\begin{aligned} x_i x_j - x_j x_i &= 0 \\ \partial_i \partial_j - \partial_j \partial_i &= 0. \\ \partial_i r - r \partial_i &= \frac{dr}{dx_i}, \text{ where } r \in R. \end{aligned}$$

Any element  $P \in D_R$  can be uniquely written as a finite sum

$$P = \sum a_{\alpha\beta} \mathbf{x}^\alpha \partial^\beta,$$



where  $a_{\alpha\beta} \in k$  and we use the multi-degree notation  $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ ,  $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$  for any  $\alpha, \beta \in \mathbb{N}^n$ . As usual, we will denote  $|\alpha| = \alpha_1 + \cdots + \alpha_n$ .

The ring of differential operators  $D_R$  is a left and right Noetherian ring and has an increasing filtration  $\{\Sigma_v\}_{v \geq 0}$  of finitely generated  $R$ -submodules satisfying  $\forall v, w \geq 0$ :

$$\begin{aligned} \cdot \bigcup_{v \in \mathbb{N}} \Sigma_v &= D_R \\ \cdot \Sigma_v \Sigma_w &= \Sigma_{v+w}. \end{aligned}$$

such that the corresponding associated graded ring

$$gr_\Sigma D_R = \Sigma_0 \oplus \frac{\Sigma_1}{\Sigma_0} \oplus \cdots$$

is isomorphic to a polynomial ring in  $2n$  variables  $k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ .

There are several ways to construct such a filtration but we will only consider the natural increasing filtration given by the order. Recall that the *order* of a differential operator  $P = \sum a_{\alpha\beta} \mathbf{x}^\alpha \partial^\beta$  is the integer  $o(P) = \sup \{|\beta| \mid a_{\alpha\beta} \neq 0\}$ . Then, the *order filtration* is given by the sets of differential operators of order less than  $v$ ,

$$\Sigma_v^{(0,1)} = \{P \in D_R \mid o(P) \leq v\}.$$

The superscript we use is just because this filtration is associated to the weight vector<sup>2</sup>  $(0, 1) \in \mathbb{Z}^{2n}$ , i.e. we set  $\deg x_i = 0$  and  $\deg \partial_i = 1$ . For simplicity we will also denote the associated graded ring by  $gr_{(0,1)} D_R$  and an explicit isomorphism with a polynomial ring in  $2n$  variables is given as follows:

The *principal symbol* of a differential operator  $P = \sum a_{\alpha\beta} \mathbf{x}^\alpha \partial^\beta \in D_R$  is the element of the polynomial ring  $k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$  in the independent variables  $\xi_1, \dots, \xi_n$  defined by:

$$\sigma_{(0,1)}(P) = \sum_{|\alpha|=o(P)} a_{\alpha\beta} \mathbf{x}^\alpha \xi^\beta,$$

where we use again the multidegree notation  $\xi^\beta := \xi_1^{\beta_1} \cdots \xi_n^{\beta_n}$ . Then, the map:

$$\begin{array}{ccc} gr_{(0,1)} D_R & \longrightarrow & k[x_1, \dots, x_n, \xi_1, \dots, \xi_n] \\ \overline{P} & \longmapsto & \sigma_{(0,1)}(P) \end{array}.$$

is an isomorphism of commutative rings.

*Warning:* At some point in these lectures we will also have to consider the ring of differential operators over the formal power series ring  $S = k[[x_1, \dots, x_n]]$ . In this case we have  $D_S = D_R \otimes_R S = k[[x_1, \dots, x_n]] \langle \partial_1, \dots, \partial_n \rangle$ , i.e. the non commutative  $S$ -algebra

---

<sup>2</sup>In general one may also consider filtrations associated to other weight vectors  $(u, v) \in \mathbb{Z}^{2n}$  with  $u + v \geq 0$ , but then the corresponding associated graded ring  $gr_{(u,v)} D_R$  is not necessarily a polynomial ring.

generated by the partial derivatives  $\partial_i = \frac{d}{dx_i}$ , with the same relations as given before. We can mimic what we did before to prove that the graded ring  $gr_{(0,1)}D_S$  associated to the order filtration on  $D_S$  is isomorphic to  $k[[x_1, \dots, x_n]][\xi_1, \dots, \xi_n]$ .

**2.2. Modules over the ring of differential operators.**  $D_R$  is a non-commutative ring so by a  $D_R$ -module we will always mean a left  $D_R$ -module. Now we will present the main examples we will consider in these lectures: the polynomial ring  $R$ , the localizations  $R_f$  at any element  $f \in R$  and the local cohomology modules  $H_I^r(R)$  where  $I \subseteq R$  is any ideal.

· **Polynomial ring  $R$ :** The action of  $x_i$  on a polynomial  $f \in R$  is just the multiplication by  $x_i$ . The action of  $\partial_i$  is the usual derivation with respect to the corresponding variable, i.e.  $\partial_i \cdot f = \frac{df}{dx_i}$ .

Notice that we have the presentation

$$R = \frac{D_R}{D_R(\partial_1, \dots, \partial_n)} = \frac{k[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle}{(\partial_1, \dots, \partial_n)}.$$

In particular,  $R$  is a finitely generated  $D_R$ -module.

· **Localizations  $R_f$ :** Consider the localization of  $R$  at any polynomial  $f \in R$

$$R_f := \left\{ \frac{g}{f^n} \mid g \in R, n \geq 0 \right\}$$

Again, the action of  $x_i$  on  $\frac{g}{f^n} \in R_f$  is the multiplication and the action of  $\partial_i$  is given by Leibniz rule.

A deep result states that  $R_f$  is the  $D_R$ -module generated by  $\frac{1}{f^\ell}$ , where  $\ell$  is the smallest integer root of the so-called *Bernstein-Sato* polynomial of  $f$ . We will skip the details on this theory since we will not use it in this work. What we highlight from this result is that we have a presentation

$$R_f = D_R \cdot \frac{1}{f^\ell} = \frac{D_R}{\text{Ann}_{D_R}(\frac{1}{f^\ell})}$$

thus  $R_f$  is a finitely generated  $D_R$ -module. More generally, given any  $D_R$ -module  $M$ , the localization  $M_f = M \otimes_R R_f$  is also a  $D_R$ -module.

· **Localizations at monomials:** When  $f$  is a monomial  $x_1 \cdots x_p$ ,  $p \leq n$  we can give a more precise description<sup>3</sup>.

$$R_{x_1 \cdots x_p} = \frac{D_R}{D_R(x_1 \partial_1 + 1, \dots, x_p \partial_p + 1, \partial_{p+1}, \dots, \partial_n)}$$

---

<sup>3</sup>Given the relation  $x_i \partial_i + 1 = 0$  one may understand  $\partial_i$  as the fraction  $\frac{1}{x_i}$  in the localization.

· **Local cohomology modules:** Let  $M$  be a  $D_R$ -module and  $I \subseteq R$  any ideal. Then, using the Čech complex  $\check{C}_I^\bullet(M)$  we can give a  $D_R$ -module structure on the local cohomology modules  $H_I^r(M)$ .

The local cohomology of the polynomial ring  $R$  with respect to the homogeneous maximal ideal  $\mathfrak{m}$  has the following presentation (see [56])

$$H_{\mathfrak{m}}^n(R) = \frac{D_R}{D_R(x_1, \dots, x_n)}$$

so it is finitely generated. It is also known that for any homogeneous prime ideal  $(x_1, \dots, x_p)$ ,  $p \leq n$  there is only a non-vanishing local cohomology modules that has the presentation

$$H_{(x_1, \dots, x_p)}^p(R) = \frac{D_R}{D_R(x_1, \dots, x_p, \partial_{p+1}, \dots, \partial_n)}$$

In the following subsection we will see that in general, the local cohomology modules  $H_I^r(R)$  and  $H_{\mathfrak{m}}^p(H_I^r(R))$  are finitely generated as  $D_R$ -modules. In fact, they are holonomic.

**2.2.1. Good filtrations.** A finitely generated  $D_R$ -module  $M$  has a *good* filtration  $\{\Gamma_k\}_{k \geq 0}$  compatible with the filtration  $\{\Sigma_v\}_{v \geq 0}$  on  $D_R$ , i.e.  $M$  has an increasing sequence of finitely generated  $R$ -submodules  $\Gamma_0 \subseteq \Gamma_1 \subseteq \dots \subseteq M$  satisfying :

$$\begin{aligned} \cdot \bigcup \Gamma_k &= M. \\ \cdot \Sigma_v \Gamma_k &\subseteq \Gamma_{v+k}. \end{aligned}$$

such that the associated graded module  $gr_{\Gamma} M = \Gamma_0 \oplus_{\Gamma_0} \Gamma_1 \oplus \dots$  is a finitely generated  $gr_{\Sigma} D_R$ -module.

Again, there are several ways to find good filtrations on a finitely generated  $D_R$ -module. When we have a presentation  $M = \frac{D_R}{L}$  the order filtration on  $D_R$  induces a good filtration on  $M$  such that the corresponding associated graded module is

$$gr_{(0,1)} M = \frac{gr_{(0,1)} D_R}{gr_{(0,1)} L} = \frac{k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]}{gr_{(0,1)} L}$$

where  $gr_{(0,1)} L = (\sigma_{(0,1)}(P) \mid P \in L)$ .

**2.3. Holonomic  $D_R$ -modules.** Let  $M$  be a finitely generated  $D_R$ -module, then we can define its dimension and multiplicity by means of its graded module  $gr_{\Gamma} M$  associated to a good filtration  $\{\Gamma_k\}_{k \geq 0}$ . Recall that  $gr_{\Gamma} M$  is finitely generated as a module over the polynomial ring  $gr_{\Sigma} D_R$  so we can use the theory of Hilbert functions to compute its dimension and multiplicity. Namely, let  $(x, \xi) = (x_1, \dots, x_n, \xi_1, \dots, \xi_n) \in k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$  be the homogeneous maximal ideal. The Hilbert series of the graded module  $gr_{\Gamma} M$ :

$$H(gr_{\Gamma} M; t) = \sum_{j \geq 0} \dim_k[(x, \xi)^j gr_{\Gamma} M / (x, \xi)^{j+1} gr_{\Gamma} M] \ t^j,$$

is of the form  $H(gr_{\Gamma}M; t) = q(t)/(1-t)^d$ , where  $q(t) \in \mathbb{Z}[t, t^{-1}]$  is such that  $q(1) \neq 0$ . The Krull dimension of  $gr_{\Gamma}M$  is  $d$  and the multiplicity of  $gr_{\Gamma}M$  is  $q(1)$ . These integers are independent of the good filtration on  $M$  and are called the dimension and the multiplicity of  $M$ . We will denote them  $d(M)$  and  $e(M)$  respectively. In the next section, we will use a geometric description of the dimension given by the so-called *characteristic variety*.

The following result is a deep theorem, proved by M. Sato, T. Kawai and M. Kashiwara in [76] (see also [62]), by using microlocal techniques. Later, O. Gabber [29] gave a purely algebraic proof:

**Theorem (Bernstein's inequality)** Let  $M$  be a non-zero finitely generated  $D_R$ -module. Then  $d(M) \geq n$ .

Now we single out the important class of  $D_R$ -modules having the minimal possible dimension.

**Definition (Holonomicity)** One says that a finitely generated  $D_R$ -module  $M$  is *holonomic* if  $M = 0$  or  $d(M) = n$ .

The class of holonomic modules has many good properties. Among them we find:

- Holonomic modules form a full abelian subcategory of the category of  $D_R$ -modules. In particular if  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is an exact sequence of  $D_R$ -modules, then  $M_2$  is holonomic if and only if  $M_1$  and  $M_3$  are both holonomic.
- $M$  is holonomic if and only if  $M$  has finite length as  $D_R$ -module.
- $M$  is holonomic if and only if  $\text{Ext}_{D_R}^i(M, D_R) = 0$  for all  $i \neq n$ .

The polynomial ring  $R$ , the localizations  $R_f$  at any element  $f \in R$  and the local cohomology modules  $H_I^r(R)$  are holonomic  $D_R$ -modules. We will check out this fact in the next subsection using the characteristic variety. We also point out that these modules are in fact regular holonomic modules in the sense of Mebkhout [63].

**2.4. The characteristic variety.** Our aim is to associate to a finitely generated  $D_R$ -module  $M$  equipped with a good filtration  $\{\Gamma_k\}_{k \geq 0}$  an invariant that provides a lot of information on this module. Since  $gr_{\Gamma}M$  is a finitely generated  $gr_{\Sigma}D_R$ -module, where  $gr_{\Sigma}D_R = k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ , we may construct the following:

- **Characteristic ideal:** Is the ideal in  $k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$  given by:

$$J_{\Sigma}(M) := \text{rad}(\text{Ann}_{gr_{\Sigma}D_R}(gr_{\Gamma}M)).$$

The characteristic ideal depends on the filtration  $\{\Sigma_v\}_{v \geq 0}$  but, once the filtration is fixed,  $J_{\Sigma}(M)$  is independent of the good filtration on  $M$ .

- **Characteristic variety:** Is the closed algebraic set given by:

$$C_\Sigma(M) := V(J_\Sigma(M)) \subseteq \text{Spec}(k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]).$$

From now on we are only going to consider the ring of differential operators with the order filtration  $\{\Sigma_v^{(0,1)}\}_{v \geq 0}$  so the characteristic ideal and the characteristic variety that we will use in this work will be denoted simply as

$$J(M) := \text{rad}(\text{Ann}_{gr_{(0,1)}D_R}(gr_\Gamma M)).$$

$$C(M) := V(J(M))$$

If the reader is interested on the behavior of the characteristic variety  $C_{(u,v)}(M)$  associated to the filtration given by a weight vector  $(u, v)$  we recommend to take a look at [79], [15]. An interesting feature is that the Krull dimension of the characteristic variety does not depend on the filtration. This provides a geometric description of the dimension of a finitely generated  $D_R$ -module. Namely, we have  $\dim C(M) = d(M)$ . In particular  $C(M) = 0$  if and only if  $M = 0$ .

When our finitely generated  $D_R$ -module has a presentation  $M = \frac{D_R}{L}$  we have a good filtration on  $M$  induced by the order filtration on  $D_R$  such that

$$gr_{(0,1)}M = \frac{k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]}{gr_{(0,1)}L}$$

Therefore, the characteristic variety is given by the ideal  $J(M) = \text{rad}(gr_{(0,1)}L)$ , where  $gr_{(0,1)}L$  is the ideal generated by the symbols  $\sigma_{(0,1)}(P)$  for all  $P \in L$ .

- **Polynomial ring  $R$ :** Consider the presentation

$$R = \frac{D_R}{D_R(\partial_1, \dots, \partial_n)}.$$

Therefore we have  $J(R) = (\xi_1, \dots, \xi_n)$

- **Localizations at monomials:** Consider the presentation

$$R_{x_1 \dots x_p} = \frac{D_R}{D_R(x_1 \partial_1 + 1, \dots, x_p \partial_p + 1, \partial_{p+1}, \dots, \partial_n)}.$$

Therefore we have  $J(R_{x_1 \dots x_p}) = (x_1 \xi_1, \dots, x_p \xi_p, \xi_{p+1}, \dots, \xi_n)$

- **Local cohomology modules:** Consider the presentation

$$H_{(x_1, \dots, x_p)}^p(R) = \frac{D_R}{D_R(x_1, \dots, x_p, \partial_{p+1}, \dots, \partial_n)}.$$

Therefore we have  $J(H_{(x_1, \dots, x_p)}^p(R)) = (x_1, \dots, x_p, \xi_{p+1}, \dots, \xi_n)$ .

In particular,  $J(H_{\mathfrak{m}}^n(R)) = (x_1, \dots, x_n)$

Notice that the characteristic variety of the polynomial ring  $R$  or the local cohomology modules  $H_{(x_1, \dots, x_p)}^p(R)$  are irreducible, but the characteristic variety of the localization  $R_{x_1 \dots x_p}$  is not. We can refine the characteristic variety with the so-called *characteristic cycle* that encodes its components with a certain multiplicity.

- **Characteristic cycle:** Is the formal sum

$$CC(M) = \sum m_i V_i$$

taken over all the irreducible components  $V_i$  of the characteristic variety  $C(M)$  and the  $m_i$ 's are the multiplicities of  $gr_\Gamma M$  at a generic point along each component  $V_i$ .

The multiplicities can also be described using the theory of Hilbert functions. Let  $V_i = V(\mathfrak{p}_i) \subseteq C(M)$  be an irreducible component, where  $\mathfrak{p}_i \in \text{Spec}(gr_{(0,1)} D_R)$ . Then  $m_i$  is the multiplicity of the module  $gr_\Gamma M_{\mathfrak{p}_i}$ . Namely, the Hilbert series:

$$H(gr_\Gamma M_{\mathfrak{p}_i}; t) = \sum_{j \geq 0} \dim_k [\mathfrak{p}_i^j gr_\Gamma M_{\mathfrak{p}_i} / \mathfrak{p}_i^{j+1} gr_\Gamma M_{\mathfrak{p}_i}] t^j$$

is in the form  $H(gr_\Gamma M_{\mathfrak{p}_i}; t) = q_i(t)/(1-t)^{d_i}$ , where  $q_i(t) \in \mathbb{Z}[t, t^{-1}]$  is such that  $q_i(1) \neq 0$ , so the multiplicity in the characteristic cycle of the irreducible component  $V_i$  is then  $m_i = q_i(1)$ .

*Warning:* If we consider the formal power series ring  $S = k[[x_1, \dots, x_n]]$  and its corresponding ring of differential operators  $D_S$  we can mimic all the above constructions. Thus we can define the class of holonomic  $D_S$ -modules and construct the corresponding characteristic cycle. The same can be done in the analytic case, i.e. when  $S = \mathbb{C}\{x_1, \dots, x_n\}$  is the ring of convergent series with complex coefficients.

However one must be careful with the components of the characteristic variety when we work over the polynomial ring  $k[x_1, \dots, x_n]$ , the formal power series ring  $k[[x_1, \dots, x_n]]$  or the analytic case  $\mathbb{C}\{x_1, \dots, x_n\}$  since they may differ. In the analytic case, F. Pham [71] (see also [48]) completely described these components. His result states that the irreducible components are conormal bundles  $T_{X_i}^* X$  relative to  $X_i \subseteq X = \mathbb{C}^n$  so we have

$$CC(M) = \sum m_i T_{X_i}^* X$$

The characteristic ideals of the examples we will use in these lectures are going to be monomial ideals so we will not have problems with their primary decomposition when viewed over the polynomial ring or over any series ring so there will be no problem borrowing the notation from the analytic case. In the sequel we will just denote

$$X = \text{Spec } k[x_1, \dots, x_n] = \mathbb{A}_k^n$$

$$T^* X = \text{Spec } k[x_1, \dots, x_n, \xi_1, \dots, \xi_n] = \mathbb{A}_k^{2n}$$

$$T_{X_\alpha}^* X = V(\{x_i \mid \alpha_i = 1\}, \{\xi_i \mid \alpha_i = 0\}) \subseteq T^* X$$

where  $X_\alpha = V(\mathfrak{p}_\alpha) \subseteq X$  is the variety defined by  $\mathfrak{p}_\alpha := \langle x_i \mid \alpha_i \neq 0 \rangle$ ,  $\alpha \in \{0, 1\}^n$ . As usual we denote  $T_X^* X = V(\xi_1, \dots, \xi_n)$  for the case  $\alpha = (0, \dots, 0) \in \{0, 1\}^n$ . This notation is very useful when we consider the projection  $\pi : T^* X \longrightarrow X$  given by the map  $\pi(\mathbf{x}, \xi) = \mathbf{x}$ , since

$$\pi(T_{X_i}^* X) = X_i$$

**2.4.1. Examples.** In general, the multiplicities of the components of the characteristic variety might be difficult to compute but, when we have a presentation  $M = \frac{D_R}{L}$  such that the ideal  $gr_{(0,1)} L$  is radical, the associated multiplicities are 1. This is what happens with the examples we are dealing with.

· **Polynomial ring  $R$ :** The characteristic variety  $C(R) = V(\xi_1, \dots, \xi_n)$  has only a component and the associated multiplicity is 1. Therefore

$$CC(R) = T_X^* X$$

· **Localizations at monomials:** The characteristic variety

$$C(R_{x_1 \dots x_p}) = V(x_1 \xi_1, \dots, x_p \xi_p, \xi_{p+1}, \dots, \xi_n)$$

has  $2^p$  components with associated multiplicity 1. Namely, we have

$$\begin{aligned} CC(R_{x_1}) &= T_X^* X + T_{X_{(1,0,\dots,0)}}^* X \\ CC(R_{x_1 x_2}) &= T_X^* X + T_{X_{(1,0,\dots,0)}}^* X + T_{X_{(0,1,0,\dots,0)}}^* X + T_{X_{(1,1,0,\dots,0)}}^* X \end{aligned}$$

and in general

$$CC(R_{x_1 x_2 \dots x_p}) = \sum_{\beta \leq \alpha} T_{X_\beta}^* X, \quad \text{where } \alpha = (\underbrace{1, \dots, 1}_p, 0, \dots, 0).$$

· **Local cohomology modules:** The characteristic variety

$$C(H_{(x_1, \dots, x_p)}^p(R)) = V(x_1, \dots, x_p, \xi_{p+1}, \dots, \xi_n)$$

has only a component and the associated multiplicity is 1. Therefore

$$CC(H_{(x_1, \dots, x_p)}^p(R)) = T_{X_\alpha}^* X, \quad \text{where } \alpha = (\underbrace{1, \dots, 1}_p, 0, \dots, 0)$$

In particular,  $CC(H_{\mathfrak{m}}^n(R)) = T_{X_1}^* X$

**2.5. Applications.** The characteristic cycle turns out to be a very useful tool in the study of  $D_R$ -modules. It is an invariant of the category of  $D_R$ -modules that also provides information on the object when viewed as  $R$ -module. Mainly, the varieties that appear in the formula  $CC(M) = \sum m_\alpha T_{X_\alpha}^* X$  for any given holonomic  $D_R$ -module  $M$  describe the support of  $M$  as  $R$ -module, but we also get some extra information coming from the corresponding multiplicities.

· **Support as  $R$ -module:** Let

$$\pi : \operatorname{Spec}(k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]) \longrightarrow \operatorname{Spec}(k[x_1, \dots, x_n])$$

be the projection map defined by  $\pi(x, \xi) = x$ . Then, for any holonomic  $D_R$ -module, we have:

$$\operatorname{Supp}_R(M) = \pi(C(M)).$$

Therefore, the notation that we use to describe the characteristic cycle will be very convenient. Namely, if  $CC(M) = \sum m_\alpha T_{X_\alpha}^* X$  then  $\operatorname{Supp}_R(M) = \bigcup X_\alpha$ .

In general, the characteristic cycle is difficult to compute directly. The following property will be very useful when computing the characteristic cycle of local cohomology modules via the Čech complex or the Mayer-Vietoris sequence.

· **Additivity of the characteristic cycle with respect to exact sequences:** Let  $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$  be an exact sequence of holonomic  $D_R$ -modules. Then, we have  $CC(M_2) = CC(M_1) + CC(M_3)$ .

When  $R = k[[x_1, \dots, x_n]]$  we can describe Lyubeznik numbers using the characteristic cycle of the local cohomology module  $H_{\mathfrak{m}}^p(H_I^{n-i}(R))$ .

· **Lyubeznik numbers:** Recall that these invariants are defined as

$$\lambda_{p,i}(R/I) = \mu_p(\mathfrak{m}, H_I^{n-i}(R)) = \mu_0(\mathfrak{m}, H_{\mathfrak{m}}^p(H_I^{n-i}(R))),$$

so we have  $H_{\mathfrak{m}}^p(H_I^{n-i}(R)) \cong E(R/\mathfrak{m})^{\lambda_{p,i}}$ . From the isomorphism  $E(R/\mathfrak{m}) \cong H_{\mathfrak{m}}^n(R)$  and the additivity of the characteristic cycle we get

$$CC(H_{\mathfrak{m}}^p(H_I^{n-i}(R))) = \lambda_{p,i} T_{X_1}^* X,$$

where  $X_1$  is the variety defined by  $\mathfrak{m} \subseteq R$ .

· **Some invariants of local rings:** A generalization of Lyubeznik numbers has been given in [3] using the characteristic cycles of local cohomology modules. The proof of the following theorem is analogous to the proof of [56, Thm. 4.1] but one must be careful with the behavior of the characteristic cycle so some results on direct images of  $D_R$ -modules are required.

**Theorem 2.1.** *Let  $A$  be a ring which admits an epimorphism  $\pi : R \longrightarrow A$ , where  $R = k[[x_1, \dots, x_n]]$  is the formal power series ring. Let  $I = \ker \pi$  and  $\mathfrak{p} \in \operatorname{Spec}(R)$  such that  $I \subseteq \mathfrak{p}$ . Consider the characteristic cycles:*



- $CC(H_I^{n-i}(R)) = \sum m_{i,\alpha} T_{X_\alpha}^* X$
- $CC(H_{\mathfrak{p}}^p(H_I^{n-i}(R))) = \sum \lambda_{\mathfrak{p},p,i,\alpha} T_{X_\alpha}^* X$

Then, the following multiplicities do not depend neither on  $R$  nor on  $\pi$ :

- The multiplicities  $m_{i,\alpha}$  only depend on  $A$ ,  $i$  and  $\alpha$ .
- The multiplicities  $\lambda_{\mathfrak{p},p,i,\alpha}$  only depend on  $A$ ,  $\mathfrak{p}$ ,  $p$ ,  $i$  and  $\alpha$ .

Among these multiplicities we may find:

- *Bass numbers*:  $\lambda_{\mathfrak{p},p,i,\alpha_{\mathfrak{p}}} = \mu_p(\mathfrak{p}, H_I^{n-i}(R))$ , where  $X_{\alpha_{\mathfrak{p}}}$  is the variety defined by  $\mathfrak{p} \subseteq R$ .
- *Lyubeznik numbers*:  $\lambda_{\mathfrak{m},p,i,\alpha_{\mathfrak{m}}} = \lambda_{p,i}(A)$ , where  $X_{\alpha_{\mathfrak{m}}}$  is the variety defined by  $\mathfrak{m} \subseteq R$ .

Collecting the multiplicities  $m_{i,\alpha}$  of the characteristic cycle of  $H_I^{n-i}(R)$  by the dimension of the corresponding varieties we get the coarser invariants:

$$\gamma_{p,i}(A) := \left\{ \sum m_{i,\alpha} \mid \dim X_\alpha = p \right\}.$$

These invariants have the same properties as Lyubeznik numbers (see [3]). Namely, let  $d = \dim A$ . Then  $\gamma_{d,d}(A) \neq 0$  and  $\gamma_{p,i}(A) = 0$  if  $i > d$ ,  $p > i$  so we can also consider the following table

$$\Gamma(A) = \begin{pmatrix} \gamma_{0,0} & \cdots & \gamma_{0,d} \\ & \ddots & \vdots \\ & & \gamma_{d,d} \end{pmatrix}.$$

Each column gives us information on the support of a local cohomology module  $H_I^{n-i}(R)$ , in particular we have  $\dim_R \text{Supp}_R H_I^{n-i}(R) = \max\{p \mid \gamma_{p,i} \neq 0\}$ .

## Lecture 2: Local cohomology modules supported on monomial ideals

Let  $R = k[x_1, \dots, x_n]$  be the polynomial ring in  $n$  independent variables, where  $k$  is a field. An ideal  $I \subseteq R$  is said to be a squarefree monomial ideal if it may be generated by squarefree monomials  $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ , where  $\alpha \in \{0, 1\}^n$ . Its minimal primary decomposition is given in terms of face ideals  $\mathfrak{p}_\alpha := \langle x_i \mid \alpha_i \neq 0 \rangle$ ,  $\alpha \in \{0, 1\}^n$ . For simplicity we will denote the homogeneous maximal ideal  $\mathfrak{m} := \mathfrak{p}_1 = (x_1, \dots, x_n)$ , where  $\mathbf{1} = (1, \dots, 1)$ . As usual, we denote  $|\alpha| = \alpha_1 + \cdots + \alpha_n$  and  $\varepsilon_1, \dots, \varepsilon_n$  will be the natural basis of  $\mathbb{Z}^n$ .

The Alexander dual ideal of  $I$  is the ideal  $I^\vee = (\mathbf{x}^\alpha \mid \mathbf{x}^{1-\alpha} \notin I)$ . The minimal primary decomposition of  $I^\vee$  can be easily described from  $I$ . Namely, let  $\{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_r}\}$  be a minimal system of generators of  $I$ . Then, the minimal primary decomposition of  $I_\Delta^\vee$  is of the form  $I_\Delta^\vee = \mathfrak{p}_{\alpha_1} \cap \cdots \cap \mathfrak{p}_{\alpha_r}$ , and we have  $I^{\vee\vee} = I$ .

The polynomial ring  $R = k[x_1, \dots, x_n]$  has a natural  $\mathbb{Z}^n$ -graduation given by  $\deg x_i = \varepsilon_i$ . The quotients  $R/I$ , where  $I \subseteq R$  is a monomial ideal, and the localizations  $R_{\mathbf{x}^\alpha}$ ,  $\alpha \in \{0, 1\}^n$  inherit a natural  $\mathbb{Z}^n$ -graded structure. Then, by using the Čech complex, the local cohomology modules  $H_{\mathfrak{m}}^r(R/I)$  and  $H_I^r(R)$  also have a  $\mathbb{Z}^n$ -graded structure.

In the last decade or so there has been a lot of progress on the understanding of this  $\mathbb{Z}^n$ -graded structure but the germ of the theory is the fundamental theorem of M. Hochster, that finally appeared in [80, Thm. II 4.1], where he gives a description of the Hilbert series of  $H_{\mathfrak{m}}^r(R/I)$ .

To describe this formula we will make use of the *Stanley-Reisner correspondence* that states that to any squarefree monomial ideal  $I \subseteq R$  one associates a simplicial complex  $\Delta$  defined over the set of vertices  $\{x_1, \dots, x_n\}$  such that  $I = I_\Delta := (\mathbf{x}^\alpha \mid \sigma_\alpha \notin \Delta)$ , where  $\sigma_\alpha := \{x_i \mid \alpha_i = 1\}$  for  $\alpha \in \{0, 1\}^n$ . We point out that this correspondence is compatible with Alexander duality in the sense that  $I_\Delta^\vee = I_{\Delta^\vee}$ , where the Alexander dual simplicial complex is  $\Delta^\vee := \{\sigma_{1-\alpha} \mid \sigma_\alpha \notin \Delta\}$ , i.e.  $\Delta^\vee$  consists of the complements of the nonfaces of  $\Delta$ .

In this section are also going to use the following subcomplexes associated to the face  $\sigma_\alpha \in \Delta$ ,  $\alpha \in \{0, 1\}^n$ :

- **restriction to  $\sigma_\alpha$ :**  $\Delta_\alpha := \{\tau \in \Delta \mid \tau \subseteq \sigma_\alpha\}$
- **link of  $\sigma_\alpha$ :**  $\text{link}_\alpha \Delta := \{\tau \in \Delta \mid \sigma_\alpha \cap \tau = \emptyset, \sigma_\alpha \cup \tau \in \Delta\}$

We have to point out that the equality of complexes  $\Delta_{1-\alpha}^\vee = (\text{link}_\alpha \Delta)^\vee$  and Alexander duality provide an isomorphism of reduced simplicial (co-)homology groups:

$$\tilde{H}_{n-|\alpha|-r-1}(\text{link}_\alpha \Delta; k) \cong \tilde{H}^{r-2}(\Delta_{1-\alpha}^\vee; k)$$

**3.1.  $\mathbb{Z}^n$ -graded structure of  $H_{\mathfrak{m}}^r(R/I)$ .** M. Hochster's formula for the  $\mathbb{Z}^n$ -graded Hilbert series of the local cohomology modules  $H_{\mathfrak{m}}^r(R/I)$  is expressed in terms of the reduced simplicial cohomology of links of the simplicial complex  $\Delta$  associated to the ideal  $I$ .

**Theorem** (Hochster). *Let  $I = I_\Delta$  be the Stanley-Reisner ideal of a simplicial complex  $\Delta$ . Then, the  $\mathbb{Z}^n$ -graded Hilbert series of  $H_{\mathfrak{m}}^r(R/I)$  is:*

$$H(H_{\mathfrak{m}}^r(R/I); \mathbf{x}) = \sum_{\sigma_\alpha \in \Delta} \dim_k \tilde{H}^{r-|\alpha|-1}(\text{link}_\alpha \Delta; k) \prod_{\alpha_i=1} \frac{x_i^{-1}}{1 - x_i^{-1}}.$$

From M. Hochster's formula we deduce the isomorphisms

$$H_{\mathfrak{m}}^r(R/I)_\beta \cong \tilde{H}^{r-|\alpha|-1}(\text{link}_\alpha \Delta; k), \quad \forall \beta \in \mathbb{Z}^n \text{ such that } \sigma_\alpha = \text{sup}_-(\beta),$$

where  $\sup_-(\beta) := \{x_i \mid \beta_i < 0\}$ . We also deduce that the multiplication by the variable  $x_i$  establishes an isomorphism between the pieces  $H_{\mathfrak{m}}^r(R/I)_{\beta}$  and  $H_{\mathfrak{m}}^r(R/I)_{\beta+\varepsilon_i}$  for all  $\beta \in \mathbb{Z}^n$  such that  $\beta_i \neq -1$ .

Notice then that, in order to determine the  $\mathbb{Z}^n$ -graded structure of this module, we only have to determine the multiplication by  $x_i$  on the pieces  $H_{\mathfrak{m}}^r(R/I)_{-\alpha}$ ,  $\alpha \in \{0, 1\}^n$ . H. G. Gräbe [36], gave a topological interpretation of these multiplications.

**Theorem** (Gräbe). *For all  $\alpha \in \{0, 1\}^n$  such that  $\sigma_{\alpha} \in \Delta$ , the morphism of multiplication by the variable  $x_i$ :*

$$\cdot x_i : H_{\mathfrak{m}}^r(R/I)_{-\alpha} \longrightarrow H_{\mathfrak{m}}^r(R/I)_{-(\alpha-\varepsilon_i)}$$

*corresponds to the morphism*

$$\tilde{H}^{r-|\alpha|-1}(\text{link}_{\alpha}\Delta; k) \longrightarrow \tilde{H}^{r-|\alpha-\varepsilon_i|-1}(\text{link}_{\alpha-\varepsilon_i}\Delta; k),$$

*or equivalently the morphism*

$$\tilde{H}^{r-2}(\Delta_{1-\alpha}^{\vee}; k) \longrightarrow \tilde{H}^{r-2}(\Delta_{1-\alpha-\varepsilon_i}^{\vee}; k),$$

*induced by the inclusion  $\Delta_{1-\alpha-\varepsilon_i}^{\vee} \subseteq \Delta_{1-\alpha}^{\vee}$ .*

**3.2.  $\mathbb{Z}^n$ -graded structure of  $H_I^r(R)$ .** Inspired by M. Hochster's formula, N. Terai [81] gave a description of the  $\mathbb{Z}^n$ -graded Hilbert series of the local cohomology modules  $H_I^r(R)$ , in this case expressed in terms of the reduced simplicial homology of the links.

**Theorem** (Terai). *Let  $I = I_{\Delta}$  be the Stanley-Reisner ideal of a simplicial complex  $\Delta$ . Then, the graded Hilbert series of  $H_I^r(R)$  is:*

$$H(H_I^r(R); \mathbf{x}) = \sum_{\alpha \in \{0, 1\}^n} \dim_k \tilde{H}_{n-r-|\alpha|-1}(\text{link}_{\alpha}\Delta; k) \prod_{\alpha_i=0} \frac{x_i^{-1}}{1-x_i^{-1}} \prod_{\alpha_j=1} \frac{1}{1-x_j}.$$

From N. Terai's formula one also may deduce the isomorphisms

$$H_I^r(R)_{\beta} \cong \tilde{H}_{n-r-|\alpha|-1}(\text{link}_{\alpha}\Delta; k), \quad \forall \beta \in \mathbb{Z}^n \text{ such that } \sigma_{\alpha} = \sup_-(\beta)$$

and that the multiplication by the variable  $x_i$  establishes an isomorphism between the pieces  $H_I^r(R)_{\beta}$  and  $H_I^r(R)_{\beta+\varepsilon_i}$  for all  $\beta \in \mathbb{Z}^n$  such that  $\beta_i \neq -1$ .

At the same time and independently, M. Mustașă [66] also described the pieces of the local cohomology modules  $H_I^r(R)$  but he also gave a topological interpretation of the multiplication by  $x_i$  on the pieces  $H_I^r(R)_{-\alpha}$ ,  $\alpha \in \{0, 1\}^n$ .

**Theorem** (Mustașă). *Let  $I = I_{\Delta}$  be the Stanley-Reisner ideal of a simplicial complex  $\Delta$ . Then,*

$$H_I^r(R)_{\beta} \cong \tilde{H}^{r-2}(\Delta_{1-\alpha}^{\vee}; k), \quad \forall \beta \in \mathbb{Z}^n \text{ such that } \sigma_{\alpha} = \sup_-(\beta)$$

*Moreover, for all  $\alpha \in \{0, 1\}^n$  such that  $\sigma_{\alpha} \in \Delta$ , the morphism of multiplication by the variable  $x_i$ :*

$$\cdot x_i : H_I^r(R)_{-\alpha} \longrightarrow H_I^r(R)_{-(\alpha-\varepsilon_i)}$$

corresponds to the morphism

$$\tilde{H}^{r-2}(\Delta_{\mathbf{1}-\alpha}^\vee; k) \longrightarrow \tilde{H}^{r-2}(\Delta_{\mathbf{1}-\alpha-\varepsilon_i}^\vee; k),$$

induced by the inclusion  $\Delta_{\mathbf{1}-\alpha-\varepsilon_i}^\vee \subseteq \Delta_{\mathbf{1}-\alpha}^\vee$ .

We remark that the formulas of M. Hochster and N. Terai are equivalent by using the Čech hull and Alexander duality (see [64]). The same happens with the formulas of H. G. Gräbe and M. Mustașă.

**3.3. A general framework: Squarefree and straight modules.** K. Yanagawa [86] introduced the notion of squarefree modules over a polynomial ring  $R = k[x_1, \dots, x_n]$  to generalize the theory of Stanley-Reisner rings. In this setting one can apply homological methods to study monomial ideals in a more systematical way. We recall now his definition

**Definition 3.1.** ([86]) *A  $\mathbb{N}^n$ -graded module  $M$  is said to be **squarefree** if the following two conditions are satisfied:*

- i)  $\dim_k M_\alpha < \infty$  for all  $\alpha \in \mathbb{Z}^n$ .
- ii) *The multiplication map  $M_\alpha \ni y \mapsto \mathbf{x}^\beta y \in M_{\alpha+\beta}$  is bijective for all  $\alpha, \beta \in \mathbb{N}^n$  with  $\text{supp}(\alpha + \beta) = \text{supp}(\alpha)$ .*

A squarefree monomial ideal  $I$  and the corresponding quotient ring  $R/I$  are squarefree modules. A free module  $R(-\alpha)$  shifted by  $\alpha \in \{0, 1\}^n$ , is also squarefree. In particular, the  $\mathbb{Z}^n$ -graded canonical module  $\omega_R = R(-\mathbf{1})$  of  $R$  is squarefree, where  $\mathbf{1} = (1, \dots, 1)$ . To describe the  $\mathbb{N}^n$ -graded structure of a squarefree module  $M$ , one only needs to describe the pieces  $M_\alpha$ ,  $\alpha \in \{0, 1\}^n$  and the multiplication maps  $x_i : M_\alpha \longrightarrow M_{\alpha+\varepsilon_i}$ .

The full subcategory of the category  ${}^*\text{Mod}(R)$  of  $\mathbb{Z}^n$ -graded  $R$ -modules which has as objects the squarefree modules will be denoted **Sq**. This is an abelian category closed by kernels, cokernels and extensions. It has enough injectives and projectives modules so one can develop all the usual tools in homological algebra. A more precise description of these objects is as follows:

- **Simple:**  $\text{Ext}_R^{|\alpha|}(R/\mathfrak{p}_\alpha, \omega_R)$  for any face ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^n$ .
- **Injective:**  $R/\mathfrak{p}_\alpha$  for any face ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^n$ .
- **Projective:**  $R(-\alpha)$ ,  $\alpha \in \{0, 1\}^n$ .

Building on the previous concept, K. Yanagawa [87] also developed a similar notion for  $\mathbb{Z}^n$ -graded modules.

**Definition 3.2.** ([87]) *A  $\mathbb{Z}^n$ -graded module  $M$  is said to be **straight** if the following two conditions are satisfied:*

- i)  $\dim_k M_\alpha < \infty$  for all  $\alpha \in \mathbb{Z}^n$ .

- ii) *The multiplication map  $M_\alpha \ni y \mapsto \mathbf{x}^\beta y \in M_{\alpha+\beta}$  is bijective for all  $\alpha, \beta \in \mathbb{Z}^n$  with  $\text{supp}(\alpha + \beta) = \text{supp}(\alpha)$ .*

The main example of straight modules are the local cohomology modules of the canonical module  $H_I^r(\omega_R)$  supported on a monomial ideal  $I \subseteq R$ . Again, in order to describe the  $\mathbb{Z}^n$ -graded structure of a straight module  $M$ , one has to describe the pieces  $M_\alpha$ ,  $\alpha \in \{0, 1\}^n$  and the multiplication maps  $x_i : M_\alpha \longrightarrow M_{\alpha+\varepsilon_i}$ .

The full subcategory of the category  ${}^*\text{Mod}(R)$  of  $\mathbb{Z}^n$ -graded  $R$ -modules which has as objects the straight modules will be denoted **Str**. This is an abelian category closed by kernels, cokernels and extensions with enough injectives and projectives described as follows:

- **Simple:**  $H_{\mathfrak{p}_\alpha}^{|\alpha|}(\omega_R)$  for any face ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^n$ .
- **Injective:**  ${}^*E(R/\mathfrak{p}_\alpha)$  for any face ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^n$ .
- **Projective:**  $R_{\mathbf{x}^\alpha}(\mathbf{1})$ ,  $\alpha \in \{0, 1\}^n$ .

A slight variation of [87, Proposition 2.12] gives a nice characterization of these modules in terms of the following filtration

**Proposition 3.3.** *An  $\mathbb{Z}^n$ -graded module  $M$  is straight if and only if there is an increasing filtration  $\{F_j\}_{0 \leq j \leq n}$  of  $M$  by  $\mathbb{Z}^n$ -graded submodules and there are integers  $m_\alpha \geq 0$  for  $\alpha \in \{0, 1\}^n$ , such that for all  $0 \leq j \leq n$  one has isomorphisms*

$$F_j/F_{j-1} \cong \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ |\alpha|=j}} (H_{\mathfrak{p}_\alpha}^j(\omega_R))^{\oplus m_\alpha}.$$

Therefore we obtain a set of short exact sequences

$$\begin{aligned} (s_1) : \quad & 0 \rightarrow F_0 \rightarrow F_1 \rightarrow F_1/F_0 \rightarrow 0 \\ (s_2) : \quad & 0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_2/F_1 \rightarrow 0 \\ & \vdots \\ (s_n) : \quad & 0 \rightarrow F_{n-1} \rightarrow M \rightarrow F_n/F_{n-1} \rightarrow 0 \end{aligned}$$

The extension classes of these exact sequences determine the structure of the straight module. Each extension class of the sequence  $(s_j)$  defines an element in  ${}^*\text{Ext}_R^1(F_j/F_{j-1}, F_{j-1})$ .

For a  $\mathbb{Z}^n$ -graded  $R$ -module  $M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_\alpha$ , we call the submodule  $\bigoplus_{\alpha \in \mathbb{N}^n} M_\alpha$  the  $\mathbb{N}^n$ -graded part of  $M$ , and denote it by  $\mathcal{N}(M)$ . If  $M$  is straight then  $\mathcal{N}(M)$  is squarefree. Conversely, for any squarefree module  $N$ , there is a unique (up to isomorphism) straight module  $\mathcal{Z}(N)$  whose  $\mathbb{N}^n$ -graded part is isomorphic to  $N$ . For example,  $\mathcal{Z}(R/\mathfrak{p}_\alpha) = {}^*$

$E(R/\mathfrak{p}_\alpha)$ . It was proved in [87, Proposition 2.7] that the functors  $\mathcal{N} : \mathbf{Str} \rightarrow \mathbf{Sq}$  and  $\mathcal{Z} : \mathbf{Sq} \rightarrow \mathbf{Str}$  establish an equivalence of categories between squarefree and straight modules. For further considerations and generalizations of this theory we recommend to take a look at [88], [89].

A generalization of squarefree and straight modules was given by E. Miller [64] (see also [65]). In his terminology positively  $\mathbf{1}$ -determined modules correspond to squarefree modules and  $\mathbf{1}$ -determined module corresponds to straight modules. In this generalized framework he introduced the *Alexander duality functors* that are closely related to Matlis duality and local duality. For the case of squarefree modules and independently, T. Römer [74] also introduced Alexander duality via the exterior algebra.

#### 4. $D$ -MODULE STRUCTURE

Recently, there has been an effort made towards effective computation of local cohomology modules by using the theory of Gröbner bases over rings of differential operators. Algorithms given by U. Walther [84] and T. Oaku and N. Takayama [69] provide a utility for such computation and are both implemented in the package `D-modules` [52] for `Macaulay 2` [37].

U. Walther's algorithm is based on the construction of the Čech complex of holonomic  $D_R$ -modules. So it is necessary to give a description of the localization  $R_f$  at a polynomial  $f \in R$ . An algorithm to compute these modules was given by T. Oaku in [68]. The main ingredient of the algorithm is the computation of the Bernstein-Sato polynomial of  $f$  which turns out to be a major bottleneck due to its complexity. For some short examples we can do the job just using the `Macaulay2` command `localCohom`.

Our first goal of this section is to compute the characteristic cycle of local cohomology modules supported on monomial ideals. Our aim is to avoid a direct computation using the additivity with respect to exact sequences. Recall that, to compute this invariant directly one needs to:

- Construct a presentation of the  $D_R$ -module  $H_I^r(R)$ ,
- Compute the characteristic ideal  $J(H_I^r(R))$ ,
- Compute the primary components of  $J(H_I^r(R))$  and its multiplicities.

As we said, we can work out the first step using the `Macaulay2` command `localCohom` for some short examples and we can also use the command `charIdeal` to compute its characteristic ideal.

**4.1. Characteristic cycle of  $H_I^r(R)$ .** Throughout this Section we will consider the polynomial ring  $R = k[x_1, \dots, x_n]$  over a field  $k$  of characteristic zero. Given an squarefree

monomial ideal  $I \subseteq R$  we want to compute the characteristic cycle of the local cohomology modules  $H_I^r(R)$ .

We want to use the additivity of characteristic cycle with respect to short exact sequences to reduce the problem to the computation of the characteristic cycles of some building blocks. If we use Čech complex these building blocks are going to be localizations of the polynomial ring  $R$  at monomials. If we use the Mayer-Vietoris sequence, then the building blocks are going to be local cohomology modules supported on face ideals. In any case, the characteristic cycle of these holonomic  $D_R$ -modules were already computed in section 2.4.

4.1.1. *Using the Čech complex.* Let  $I = (f_1, \dots, f_s) \subseteq R$ , be a squarefree monomial ideal, i.e. the generators  $f_i$  are monomials of the form  $\mathbf{x}^\beta$ ,  $\beta \in \{0, 1\}^n$ . Consider the Čech complex

$$\check{C}_I^\bullet(R): 0 \longrightarrow R \xrightarrow{d_0} \bigoplus_{1 \leq i \leq s} R_{f_i} \xrightarrow{d_1} \dots \xrightarrow{d_{s-1}} R_{f_1 \dots f_s} \longrightarrow 0$$

where the differentials  $d_p$  are defined by using the canonical localization morphism on every component. This is a complex of  $\mathbb{Z}^n$ -graded modules so the differentials can be described using the so-called *monomial matrices* introduced by E. Miller [64].

**Example:** Let  $I = (f_1, f_2, f_3) \subseteq R$ , be a squarefree monomial ideal. Then, the Čech complex can be described as

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{array}{c} R_{f_1} \\ \oplus \\ R_{f_2} \\ \oplus \\ R_{f_3} \end{array} \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}} \begin{array}{c} R_{f_1 f_2} \\ \oplus \\ R_{f_1 f_3} \\ \oplus \\ R_{f_2 f_3} \end{array} \xrightarrow{(-1, 1, -1)} R_{f_1 f_2 f_3} \longrightarrow 0$$

The source and the target of these monomial matrices are labelled by the monomials  $f_1, f_2, f_3$  and their products.

Our aim is to use these monomial matrices to construct a complex of  $k$ -vector spaces associated to each possible component  $T_{X_\alpha}^* X$ ,  $\alpha \in \{0, 1\}^n$  of the characteristic cycle such that the corresponding homology groups describe the corresponding multiplicity. To illustrate this computations we present the following:

**Example:** Consider the ideal  $I = (x_1 x_2, x_1 x_3, x_2 x_3)$  in  $R = k[x_1, x_2, x_3]$ . We have the Čech complex  $\check{C}_I^\bullet(R)$ :

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{array}{c} R_{x_1 x_2} \\ \oplus \\ R_{x_1 x_3} \\ \oplus \\ R_{x_2 x_3} \end{array} \xrightarrow{\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}} \begin{array}{c} R_{x_1 x_2 x_3} \\ \oplus \\ R_{x_1 x_2 x_3} \\ \oplus \\ R_{x_1 x_2 x_3} \end{array} \xrightarrow{(-1, 1, -1)} R_{x_1 x_2 x_3} \longrightarrow 0$$

The interested reader should try to figure out the labels for the source and target of the monomial matrices.

To compute the characteristic cycle of the local cohomology modules  $H_I^r(R)$  we have to split the Čech complex into short exact sequences and use the additivity property but we can do all in once just keeping track of any component  $T_{X_\alpha}^* X$ ,  $\alpha \in \{0, 1\}^3$  that appear in the characteristic cycle of the localizations in the Čech complex.

- For  $\alpha = (0, 0, 0)$  the component  $T_X^* X$  appears in the characteristic cycle of every localization. We illustrate this fact in the following diagram

$$\begin{array}{ccccccc}
 & & T_X^* X & & T_X^* X & & \\
 & & + & & + & & \\
 0 & \longrightarrow & T_X^* X & \longrightarrow & T_X^* X & \longrightarrow & T_X^* X \longrightarrow 0 \\
 & & + & & + & & \\
 & & T_X^* X & & T_X^* X & & 
 \end{array}$$

One can check out that the multiplicity of  $T_X^* X$  in the characteristic cycle of  $H_I^r(R)$  is the dimension of the  $r$ -th cohomology groups of the following complex of  $k$ -vector spaces that we can construct using the monomial matrices describing the Čech complex<sup>4</sup>.

$$\begin{array}{ccccccc}
 & & \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} & & \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} & & \\
 0 & \longrightarrow & k & \longrightarrow & k & \xrightarrow{(-1, 1, -1)} & k \longrightarrow 0 \\
 & & \oplus & & \oplus & & \\
 & & k & & k & & 
 \end{array}$$

This complex is acyclic so  $T_X^* X$  is not a summand of the characteristic cycle of  $H_I^r(R)$  for any  $r$ .

- For  $\alpha = (1, 1, 0)$  the component  $T_{X_{(1,1,0)}}^* X$  shows up in the following places

$$\begin{array}{ccccccc}
 & & T_{X_{(1,1,0)}}^* X & & T_{X_{(1,1,0)}}^* X & & \\
 & & + & & + & & \\
 0 & \longrightarrow & 0 & \longrightarrow & T_{X_{(1,1,0)}}^* X & \longrightarrow & T_{X_{(1,1,0)}}^* X \longrightarrow 0 \\
 & & + & & + & & \\
 & & 0 & & T_{X_{(1,1,0)}}^* X & & 
 \end{array}$$

---

<sup>4</sup>One has to interpret the non-zero entries in the matrix as inclusions of the corresponding components of the Čech complex



The complex of  $k$ -vector spaces that we obtain in this case is

$$0 \longrightarrow 0 \longrightarrow k \xrightarrow{\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}} k^3 \xrightarrow{(-1,1,-1)} k \longrightarrow 0$$

so  $CC(H_I^2(R))$  contains  $T_{X_{(1,1,0)}}^* X$  with multiplicity 1. The cases  $\alpha = (0, 1, 1), (1, 0, 1)$  are analogous.

- For  $\alpha = (1, 1, 1)$  the component  $T_{X_{(1,1,1)}}^* X$  appears in the following places

$$\begin{array}{ccccccc} & & 0 & & T_{X_{(1,1,1)}}^* X & & \\ & & + & & + & & \\ 0 & \longrightarrow & 0 & \longrightarrow & T_{X_{(1,1,1)}}^* X & \longrightarrow & T_{X_{(1,1,1)}}^* X \longrightarrow 0 \\ & & + & & + & & \\ & & 0 & & T_{X_{(1,1,1)}}^* X & & \end{array}$$

The complex of  $k$ -vector spaces that we obtain in this case is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow k^3 \xrightarrow{(-1,1,-1)} k \longrightarrow 0$$

so  $CC(H_I^2(R))$  contains  $T_{X_{(1,1,1)}}^* X$  with multiplicity 2.

Therefore there is a local cohomology module different from zero and its characteristic cycle is

$$CC(H_I^2(R)) = T_{X_{(1,1,0)}}^* X + T_{X_{(1,0,1)}}^* X + T_{X_{(0,1,1)}}^* X + 2 T_{X_{(1,1,1)}}^* X.$$

Now we consider the general case where  $I = (f_1, \dots, f_s) \subseteq R$ , is a squarefree monomial ideal and the generators  $f_i$  are monomials of the form  $\mathbf{x}^\beta$ ,  $\beta \in \{0, 1\}^n$ . Consider the Čech complex

$$\check{C}_I^\bullet(R): 0 \longrightarrow R \xrightarrow{d_0} \oplus_{1 \leq i \leq s} R_{f_i} \xrightarrow{d_1} \dots \xrightarrow{d_{s-1}} R_{f_1 \dots f_s} \longrightarrow 0.$$

with the corresponding monomial matrices that describe the morphisms in the complex. Then, for any  $\alpha \in \{0, 1\}^n$  we construct a complex of  $k$ -vector spaces  $[\check{C}_I^\bullet(R)]_\alpha$  that encodes when the component  $T_{X_\alpha}^* X$  appears in the characteristic cycle of the localizations in the Čech complex, i.e. we have a copy of the field  $k$  in each position where the component  $T_{X_\alpha}^* X$  appear. The morphisms of this complex are given by the monomial matrices that describe the Čech complex. Recall that the characteristic cycle of the localization of  $R$  at a monomial  $\mathbf{x}^\beta$  is

$$CC(R_{\mathbf{x}^\beta}) = \sum_{\alpha \leq \beta} T_{X_\alpha}^* X$$

Then:

• For  $\alpha = \mathbf{0} = (0, \dots, 0) \in \{0, 1\}^n$ , since every localization  $R_{\mathbf{x}^\beta}$  contains the component  $T_{\mathbf{x}}^* X$ , the complex of  $k$ -vector spaces associated to the Čech complex  $\check{C}_I^\bullet$  is:

$$[\check{C}_I^\bullet(R)]_{\mathbf{0}}: 0 \longrightarrow k \xrightarrow{d_0} k^s \xrightarrow{d_1} \dots \xrightarrow{d_{s-1}} k \longrightarrow 0.$$

This complex may be identified with the augmented relative simplicial cochain complex  $\tilde{\mathcal{C}}^\bullet(\Delta_s; k)$ , where  $\Delta_s$  is the full simplicial complex whose vertices  $\{x_1, \dots, x_s\}$  are labelled by the minimal system of generators of  $I$ .

• In general, for any  $\alpha \in \{0, 1\}^n$ , the component  $T_{\mathbf{x}^\alpha}^* X$  only appears in the localizations  $R_{\mathbf{x}^\beta}$  such that  $\beta \geq \alpha$  so the complex we construct is a subcomplex of  $[\check{C}_I^\bullet(R)]_{\mathbf{0}}$  and the morphisms are the corresponding restrictions. In order to give a topological interpretation, notice that, from the augmented relative simplicial chain complex  $\tilde{\mathcal{C}}_\bullet(\Delta_s; k)$ , we are taking out the pieces corresponding to the faces

$$\sigma_{\mathbf{1}-\beta} := \{x_1, \dots, x_s\} \setminus \{x_i \mid \beta_i = 1\} \in \Delta_s \text{ such that } \beta \not\geq \alpha.$$

Let  $T_\alpha := \{\sigma_{\mathbf{1}-\beta} \in \Delta_s \mid \beta \not\geq \alpha\}$  be a simplicial subcomplex of  $\Delta_s$ . Then, the complex  $[\check{C}_I^\bullet(R)]_\alpha$  may be identified with the augmented relative simplicial chain complex  $\tilde{\mathcal{C}}_\bullet(\Delta_s, T_\alpha; k)$  associated to the pair  $(\Delta_s, T_\alpha)$ . By taking homology, the multiplicity of the component  $T_{\mathbf{x}^\alpha}^* X$  in the characteristic cycle of the local cohomology modules  $H_I^r(R)$  are:

$$m_{n-r, \alpha} = \dim \tilde{H}_{r-1}(\Delta_s, T_\alpha; k) = \dim \tilde{H}_{r-2}(T_\alpha; k),$$

where the last assertion comes from the fact that  $\Delta_s$  is contractible.

**Proposition 4.1.** *Let  $I = (f_1, \dots, f_s) \subseteq R$  be a squarefree monomial ideal. Then, the characteristic cycle of the local cohomology modules  $H_I^r(R)$  is*

$$CC(H_I^r(R)) = \sum m_{n-r, \alpha} T_{\mathbf{x}^\alpha}^* X$$

where  $m_{n-r, \alpha} = \dim \tilde{H}_{r-2}(T_\alpha; k)$  and  $T_\alpha := \{\sigma_{\mathbf{1}-\beta} \in \Delta_s \mid \beta \not\geq \alpha\} \subseteq \Delta_s$ .

**Remark 4.2.** *The proof of this proposition follows from the additivity of the characteristic cycle and the fact that the monomial matrices that we use to construct the complex of  $k$ -vector spaces describe the inclusions between the localizations in the Čech complex.*

Using the techniques we will develop in section 4.3 one has a more direct proof since we have an isomorphism of complexes of  $k$ -vector spaces  $[\check{C}_I^\bullet(R)]_\alpha = \text{Hom}_{D_R}(\check{C}_I^\bullet(R), E_\alpha)$ . We recommend the reader to go back to this point after getting familiar with the theory of  $n$ -hypercubes.

4.1.2. *Using the Mayer-Vietoris sequence.* The usual method to compute local cohomology modules  $H_I^r(R)$  is to find a representation of the ideal  $I = U \cap V$  as the intersection of two simpler ideals  $U$  and  $V$  and then apply the Mayer-Vietoris sequence

$$\dots \longrightarrow H_{U+V}^r(R) \longrightarrow H_U^r(R) \oplus H_V^r(R) \longrightarrow H_{U \cap V}^r(R) \longrightarrow H_{U+V}^{r+1}(R) \longrightarrow \dots$$

In general, there are several choices for such a representation but in the case of squarefree monomial ideals we can use the good properties of a primary decomposition  $I = I_1 \cap \dots \cap I_m$  and develop a method that will allow us to study these local cohomology modules in a systematical way. This is the approach used in [1] but we have to point out that we do not obtain a closed formula for the characteristic cycle of the local cohomology modules as the one we obtained in the previous subsection. The formula obtained in [1] comes after applying an algorithm that describes this iterated Mayer-Vietoris process.

We illustrate the method with the same example we used before

**Example:** Let  $I = I_1 \cap I_2 \cap I_3 = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3)$  be the minimal primary decomposition of the squarefree monomial ideal  $I = (x_1x_2, x_1x_3, x_2x_3)$  in  $R = k[x_1, x_2, x_3]$ . To study the local cohomology modules  $H_I^r(R)$  we first use a Mayer-Vietoris sequence with:

$$\begin{aligned} U &= I_1 \cap I_2, & U \cap V &= I = I_1 \cap I_2 \cap I_3, \\ V &= I_3, & U + V &= (I_1 \cap I_2) + I_3. \end{aligned}$$

We get the long exact sequence:

$$\dots \longrightarrow H_{I_1 \cap I_2}^r(R) \oplus H_{I_3}^r(R) \longrightarrow H_I^r(R) \longrightarrow H_{(I_1 \cap I_2) + I_3}^{r+1}(R) \longrightarrow \dots$$

- The ideal  $I_1 \cap I_2 = (x_1, x_2x_3)$  is not a face ideal but we can describe the modules  $H_{I_1 \cap I_2}^r(R)$  by using a Mayer-Vietoris sequence with:

$$\begin{aligned} U &= I_1, & U \cap V &= I_1 \cap I_2, \\ V &= I_2, & U + V &= I_1 + I_2. \end{aligned}$$

- In general, not for this example, the ideal  $(I_1 \cap I_2) + I_3$  is not a face ideal but we can describe the modules  $H_{(I_1 \cap I_2) + I_3}^r(R)$  by using a Mayer-Vietoris sequence with:

$$\begin{aligned} U &= I_1 + I_3, & U \cap V &= (I_1 \cap I_2) + I_3, \\ V &= I_2 + I_3, & U + V &= I_1 + I_2 + I_3. \end{aligned}$$

We can reflect the above process in the following diagram:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
 & & H_{I_1+I_2}^{r+1}(R) & & H_{I_1+I_2+I_3}^{r+2}(R) & & \\
 & & \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & H_{I_1 \cap I_2}^r(R) \oplus H_{I_3}^r(R) & \longrightarrow & H_I^r(R) & \longrightarrow & H_{(I_1 \cap I_2) + I_3}^{r+1}(R) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & H_{I_1}^r(R) \oplus H_{I_2}^r(R) & & H_{I_1+I_3}^{r+1}(R) \oplus H_{I_2+I_3}^{r+1}(R) & & \\
 & & \uparrow & & \uparrow & & \\
 & & H_{I_1+I_2}^r(R) & & H_{I_1+I_2+I_3}^{r+1}(R) & & \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & 
 \end{array}$$

Thus, in order to describe the local cohomology modules  $H_I^r(R)$ , we have to study the modules:

$$\begin{array}{lll}
 H_{I_1}^r(R), & H_{I_1+I_2}^r(R), & H_{I_1+I_2+I_3}^r(R). \\
 H_{I_2}^r(R), & H_{I_1+I_3}^r(R) & \\
 H_{I_3}^r(R) & H_{I_2+I_3}^r(R), & 
 \end{array}$$

and the homomorphisms of the corresponding Mayer-Vietoris sequences. These modules are the local cohomology modules supported on all the ideals we can construct as sums of face ideals in the minimal primary decomposition of  $I$ . We state that these are the *initial pieces* that allow us to describe the modules  $H_I^r(R)$ . These sums of face ideals are again face ideals so they only have a non-vanishing local cohomology module. In our example, we have

$$I_1 + I_2 = I_1 + I_3 = I_2 + I_3 = I_1 + I_2 + I_3 = (x_1, x_2, x_3),$$

so the non-vanishing modules in this Mayer-Vietoris process are

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \uparrow & & & & \\
 & & H_{I_1+I_2}^3(R) & & & & \\
 & & \uparrow & & & & \\
 0 & \longrightarrow & H_{I_1 \cap I_2}^2(R) \oplus H_{I_3}^2(R) & \longrightarrow & H_I^2(R) & \longrightarrow & H_{(I_1 \cap I_2) + I_3}^3(R) \longrightarrow 0 \\
 & & \uparrow & & & & \uparrow \\
 & & H_{I_1}^2(R) \oplus H_{I_2}^2(R) & & & & H_{I_1+I_3}^3(R) \oplus H_{I_2+I_3}^3(R) \\
 & & \uparrow & & & & \uparrow \\
 & & 0 & & & & H_{I_1+I_2+I_3}^3(R) \\
 & & & & & & \uparrow \\
 & & & & & & 0
 \end{array}$$

By the additivity of the characteristic cycle we get:

$$\begin{aligned}
 CC(H_I^2(R)) &= CC(H_{(x_1, x_2)}^2(R)) + CC(H_{(x_1, x_3)}^2(R)) + CC(H_{(x_2, x_3)}^2(R)) + \\
 &\quad + 2 \ CC(H_{(x_1, x_2, x_3)}^3(R)) = \\
 &= T_{X_{(1,1,0)}}^* X + T_{X_{(1,0,1)}}^* X + T_{X_{(0,1,1)}}^* X + 2 \ T_{X_{(1,1,1)}}^* X.
 \end{aligned}$$

The general case is developed in [1] and, since it is a lot more involved, we will skip the details. We will only point out that, given the minimal primary decomposition  $I = I_1 \cap \dots \cap I_m$  of a squarefree monomial ideal, the initial pieces that describe the Mayer-Vietoris process are the local cohomology modules supported on all the sums of face ideals in the minimal primary decomposition. The key point in the whole process is to control the kernels and cokernels that appear when we split all the Mayer-Vietoris sequences into short exact sequences.

**4.1.3. Using the Mayer-Vietoris spectral sequence.** All the Mayer-Vietoris process described above can be done all in once with the help of a spectral sequence introduced in [5] and developed in [60]. We do not assume the reader to have experience with the use of spectral sequence so we will skip the construction of this one and the meaning of *degeneration at the  $E_2$ -term* that leads to a closed formula for the characteristic cycle of local cohomology modules. For those that want to get more insight on this useful tool we recommend to take a look at a good homological algebra book for the basics on spectral sequences and then go to the details for this particular case in [5] or [60].

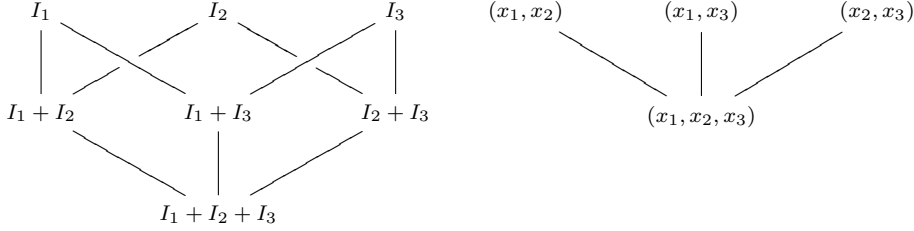
Let  $I = I_1 \cap \cdots \cap I_m$  be a minimal primary decomposition of a squarefree monomial ideal  $I \subseteq R$ . Then we have the spectral sequence:

$$E_1^{-i,j} = \bigoplus_{1 \leq \ell_1 < \cdots < \ell_i \leq m} H_{I_{\ell_1} + \cdots + I_{\ell_i}}^j(R) \implies H_I^{j-i}(R).$$

The  $E_1$ -page encodes the information given by the initial pieces we considered in the previous section. The  $E_2$ -page is more sophisticated and we need to introduce some notation.

The ideal  $I$  can be thought as the defining ideal of an arrangement  $\mathcal{A}$  of linear varieties<sup>5</sup>. It defines a poset  $P(\mathcal{A})$  formed by the intersections of the irreducible components of  $X$  and the order given by the inclusion, i.e.  $P(\mathcal{A})$  is nothing but the poset formed by all the sums of face ideals in the minimal primary decomposition of  $I$  ordered by reverse inclusion but notice that we have to identify these sums when they describe the same ideal.

**Example:** Consider the ideal  $I = I_1 \cap I_2 \cap I_3 = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3)$  in  $k[x_1, x_2, x_3]$ . The initial pieces that allowed to develop the Mayer-Vietoris process in the previous subsection can be encoded in the left poset. The poset  $P(\mathcal{A})$  we have to consider now is the one on the right



The Mayer-Vietoris spectral sequence has the following description when we consider the  $E_2$ -page:

$$E_2^{-i,j} = \varinjlim_{P(\mathcal{A})}^{(i)} H_{I_p}^j(R) \Rightarrow H_I^{j-i}(R)$$

where  $p$  is an element of the poset  $P(\mathcal{A})$ ,  $I_p$  is the defining (radical) ideal of the irreducible variety corresponding to  $p$ ,  $h(p)$  denotes the height of the ideal  $I_p$  and  $\varinjlim_{P(\mathcal{A})}^{(i)}$  is the  $i$ -th left derived functor of the direct limit functor in the category of direct systems indexed by the poset  $P(\mathcal{A})$ .

On paper the  $E_2$ -page looks more difficult to deal with but we can give a nice topological interpretation. First recall that to any poset one can associate a simplicial complex which has as vertices the elements of the poset and where a set of vertices  $p_0, \dots, p_r$  determines a  $r$ -dimensional simplex if  $p_0 < \cdots < p_r$ . Now we define  $K(> p)$  to be the simplicial complex attached to the subposet  $\{q \in P(\mathcal{A}) \mid q > p\}$  of our  $P(\mathcal{A})$ . Then, there are

<sup>5</sup>We consider this point of view since the same results will be true if we consider the defining ideal of any arrangement of linear subspaces.

$R$ -module isomorphisms

$$\varinjlim_{P(\mathcal{A})}^{(i)} H_{I_p}^j(R) \simeq \bigoplus_{h(p)=j} [H_{I_p}^j(R) \otimes_k \tilde{H}_{i-1}(K(>p); k)],$$

where  $\tilde{H}(-; k)$  denotes reduced simplicial homology. Here we agree that the reduced homology with coefficients in  $k$  of the empty simplicial complex is  $k$  in degree  $-1$  and zero otherwise.

The main result of this section is the following

**Theorem 4.3.** *Let  $\mathcal{A}$  be an arrangement of linear varieties defined by a squarefree monomial ideal  $I = I_1 \cap \cdots \cap I_m \subset R$ . Then, the Mayer–Vietoris spectral sequence*

$$E_1^{-i,j} = \bigoplus_{1 \leq \ell_1 < \cdots < \ell_i \leq m} H_{I_{\ell_1} + \cdots + I_{\ell_i}}^j(R) \implies H_I^{j-i}(R).$$

*degenerates at the  $E_2$ -page.*

The main ingredient of the proof is the fact that local cohomology modules supported on a face ideal only have one associated prime, in particular its characteristic variety only has one component. The degeneration of the Mayer–Vietoris spectral sequence provides a filtration of the local cohomology modules, where the successive quotients are given by the  $E_2$ -term.

**Corollary 4.4.** *Let  $\mathcal{A}$  be an arrangement of linear varieties defined by a squarefree monomial ideal  $I = I_1 \cap \cdots \cap I_n \subset R$ . Then, for all  $r \geq 0$  there is a filtration  $\{F_j^r\}_{0 \leq j \leq n}$  of  $H_I^r(R)$  by  $R$ -submodules such that*

$$F_j^r / F_{j-1}^r \cong \bigoplus_{h(p)=j} [H_{I_p}^j(R) \otimes_k \tilde{H}_{h(p)-r-1}(K(>p); k)].$$

Moreover, if  $\text{char } k = 0$  it is a filtration by holonomic  $D_R$ -modules. Thus we can compute the characteristic cycle of the modules  $H_I^r(R)$  from the short exact sequences

$$\begin{aligned} (s_r) : \quad & 0 \rightarrow F_{r-1}^r \rightarrow F_r^r \rightarrow F_r^r / F_{r-1}^r \rightarrow 0 \\ (s_{r+1}) : \quad & 0 \rightarrow F_r^r \rightarrow F_{r+1}^r \rightarrow F_{r+1}^r / F_r^r \rightarrow 0 \\ & \vdots \\ (s_n) : \quad & 0 \rightarrow F_{n-1}^r \rightarrow H_I^r(R) \rightarrow F_n^r / F_{n-1}^r \rightarrow 0 \end{aligned}$$

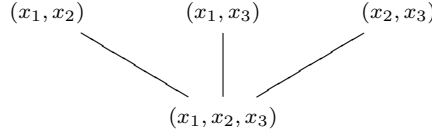
given by the filtration and the additivity of the characteristic cycle. By general properties of local cohomology modules we have  $F_j^r = 0 \quad \forall j < r$ .

**Corollary 4.5.** *The characteristic cycle of  $H_I^r(R)$  is*

$$CC(H_I^r(R)) = \sum m_{n-r,p} T_{X_p}^* X,$$

where  $m_{n-r,p} = \dim_k \tilde{H}_{h(p)-r-1}(K(>p); k)$ .

**Example:** Consider  $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) \subseteq k[x_1, x_2, x_3]$ . We have the poset



Then:

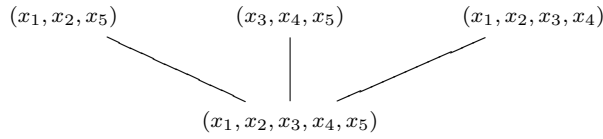
$p$	$I_p$	$K(> p)$	$\dim_k \tilde{H}_{-1}$	$\dim_k \tilde{H}_0$	$\dim_k \tilde{H}_1$
$p_1$	$(x_1, x_2)$	$\emptyset$	1	-	-
$p_2$	$(x_1, x_3)$	$\emptyset$	1	-	-
$p_3$	$(x_2, x_3)$	$\emptyset$	1	-	-
$q$	$(x_1, x_2, x_3)$	$\bullet \bullet \bullet$	-	2	-

And the characteristic cycle is

$$CC(H_I^2(R)) = T_{X_{(1,1,0)}}^* X + T_{X_{(1,0,1)}}^* X + T_{X_{(0,1,1)}}^* X + 2 T_{X_{(1,1,1)}}^* X.$$

The poset associated to the next example has the same shape as in the previous one but, since the formula also depends on the height of the ideals in the poset, the corresponding characteristic cycle is quite different.

**Example:** Consider  $I = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4) \subseteq k[x_1, x_2, x_3, x_4, x_5]$ . The poset associated to this ideal is



Therefore:

$p$	$I_p$	$K(> p)$	$\dim_k \tilde{H}_{-1}$	$\dim_k \tilde{H}_0$	$\dim_k \tilde{H}_1$
$p_1$	$(x_1, x_2, x_5)$	$\emptyset$	1	-	-
$p_2$	$(x_3, x_4, x_5)$	$\emptyset$	1	-	-
$p_3$	$(x_1, x_2, x_3, x_4)$	$\emptyset$	1	-	-
$q$	$(x_1, x_2, x_3, x_4, x_5)$	$\bullet \bullet \bullet$	-	2	-

In this case we have two local cohomology modules different from zero and their characteristic cycle is

$$CC(H_I^3(R)) = T_{X_{(1,1,0,0,1)}}^* X + T_{X_{(0,0,1,1,1)}}^* X.$$

$$CC(H_I^4(R)) = T_{X_{(1,1,1,1,0)}}^* X + 2 T_{X_{(1,1,1,1,1)}}^* X.$$

**4.2. Extracting some information.** We have seen how to compute the characteristic cycle of the local cohomology module  $H_I^r(R)$  using different techniques. This invariant describes the support of  $H_I^r(R)$  but we also get some extra information given by the



multiplicities. In this section we will see how to extract information from the support but also, how the multiplicities can describe some arithmetic properties of the quotient ring  $R/I$ .

4.2.1. *Support of local cohomology modules.* Once we know the characteristic cycle we can answer some of the questions raised by C. Huneke. In particular we can deal with:

- Annihilation of local cohomology modules:
- Cohomological dimension.
- Description of the support of local cohomology modules.
- Krull dimension of local cohomology modules.
- Artinianity of local cohomology modules.

In the following example we can see how to read all this information from the coarser invariant given by the table  $\Gamma(R/I)$  introduced in section 2.5

**Example:** Consider  $I = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4) \subseteq k[x_1, x_2, x_3, x_4, x_5]$ . We have

$$\begin{aligned} CC(H_I^3(R)) &= T_{X_{(1,1,0,0,1)}}^* X + T_{X_{(0,0,1,1,1)}}^* X. \\ CC(H_I^4(R)) &= T_{X_{(1,1,1,1,0)}}^* X + 2 T_{X_{(1,1,1,1,1)}}^* X. \end{aligned}$$

Collecting the components by their dimension we get the table  $\Gamma(R/I) = \begin{pmatrix} 0 & 2 & 0 \\ & 1 & 0 \\ & & 2 \end{pmatrix}$

- $\text{Supp}(H_I^3(R)) = V(x_1, x_2, x_5) \cup V(x_3, x_4, x_5)$  so  $\dim_R \text{Supp}_R H_I^3(R) = 2$
- $\text{Supp}(H_I^4(R)) = V(x_1, x_2, x_3, x_4)$  so  $\dim_R \text{Supp}_R H_I^4(R) = 1$
- $\text{cd}(R, I) = 4$ .

Notice that each column gives information on a different local cohomology module, in this case  $H_I^5(R) = 0$ ,  $H_I^4(R)$  and  $H_I^3(R)$  respectively. Also, each row describe the dimension of the components of the characteristic cycle. Namely, the top row describe the components of dimension zero and so on.

4.2.2. *Arithmetic properties.* The multiplicities of the characteristic cycle also provide a good test for the arithmetical properties of the quotient rings  $R/I$ .

· *Cohen-Macaulay property:* For a squarefree monomial ideal it is equivalent to have just one local cohomology module different from zero [53]. Therefore we only have to check out whether  $\Gamma(R/I)$  has just one column.

· *Buchsbaum property:* By [80, Theorem 8.1], the Buchsbaum property of  $R/I$  is equivalent to the Cohen-Macaulayness of the localized rings  $(R/I)_{\mathfrak{p}}$  for any prime ideal  $\mathfrak{p} \neq \mathfrak{m}$ . It

means that the local cohomology modules  $H_I^r(R)$  have dimension zero when  $r \neq \text{ht } I$ , i.e. these modules are Artinian. Therefore, we have to check out that the non-vanishing entries of  $\Gamma(R/I)$  are in the last column and the first row.

· *Gorenstein property*: This property is more involved and we have to check out the multiplicities. Namely, let  $I = \mathfrak{p}_{\alpha_1} \cap \cdots \cap \mathfrak{p}_{\alpha_m}$  be the minimal primary decomposition of our squarefree monomial ideal. Then,  $R/I$  is Gorenstein if and only if  $R/I$  is Cohen-Macaulay and  $m_{n-\text{ht } I, \alpha} = 1$  for all  $\alpha \geq \alpha_j$ ,  $j = 1, \dots, m$ .

**Example:** Consider the ideals in  $R = k[x_1, x_2, x_3, x_4]$ :

- $I_1 = (x_1, x_2) \cap (x_3, x_4)$ .
- $I_2 = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_2, x_4)$ .
- $I_3 = (x_1, x_2) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_3, x_4)$ .

If we compute the corresponding characteristic cycles we get:

$$\Gamma(R/I_1) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}, \quad \Gamma(R/I_2) = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 4 \\ & & 4 \end{pmatrix}, \quad \Gamma(R/I_3) = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 4 \\ & & 4 \end{pmatrix}.$$

We have:

- $R/I_1$  is Buchsbaum but it is not Cohen-Macaulay.
- $R/I_2$  is Cohen-Macaulay but it is not Gorenstein.
- $R/I_3$  is Gorenstein.

If we take a look at the multiplicities we get:

$$\begin{aligned} CC(H_{I_2}^2(R)) &= T_{X_{(1,1,0,0)}}^* X + T_{X_{(1,0,0,1)}}^* X + T_{X_{(0,1,1,0)}}^* X + T_{X_{(0,1,0,1)}}^* X + \\ &+ T_{X_{(1,1,1,0)}}^* X + 2 T_{X_{(1,1,0,1)}}^* X + T_{X_{(0,1,1,1)}}^* X + T_{X_{(1,1,1,1)}}^* X. \end{aligned}$$

and

$$\begin{aligned} CC(H_{I_3}^2(R)) &= T_{X_{(1,1,0,0)}}^* X + T_{X_{(1,0,0,1)}}^* X + T_{X_{(0,1,1,0)}}^* X + T_{X_{(0,0,1,1)}}^* X + \\ &+ T_{X_{(1,1,0,1)}}^* X + T_{X_{(1,1,1,0)}}^* X + T_{X_{(1,0,1,1)}}^* X + T_{X_{(0,1,1,1)}}^* X + \\ &+ T_{X_{(1,1,1,1)}}^* X. \end{aligned}$$

**4.2.3. Betti numbers of complements of arrangements.** Let  $\mathcal{A}$  be an arrangement of linear varieties defined by a squarefree monomial ideal  $I \subset R$ . A formula for the Betti numbers

of the complement  $\mathbb{A}_{\mathbb{R}}^n - \mathcal{A}$  has been given by Goresky–MacPherson ([34, III.1.3. Theorem A]), which states (slightly reformulated) that

$$\tilde{H}_r(\mathbb{A}_{\mathbb{R}}^n - X; \mathbb{Z}) \cong \bigoplus_p \tilde{H}^{h(p)-r-2}(K(> p); \mathbb{Z}).$$

When we work over a field, these reduced simplicial cohomology groups allowed us to compute the characteristic cycle of the local cohomology modules so if we have

$$CC(H_I^r(R)) = \sum m_{n-r,p} T_{X_p}^* X$$

then, if  $k = \mathbb{R}$  is the field of real numbers, the Betti numbers of the complement of the arrangement  $\mathcal{A}$  in  $X = \mathbb{A}_{\mathbb{R}}^n$  can be computed in terms of the multiplicities  $\{m_{n-r,p}\}$  as

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}_{\mathbb{R}}^n - \mathcal{A}; \mathbb{Q}) = \sum_p m_{n-(r+1),p}.$$

If  $k = \mathbb{C}$  is the field of complex numbers, then one has

$$\dim_{\mathbb{Q}} \tilde{H}_r(\mathbb{A}_{\mathbb{C}}^n - \mathcal{A}; \mathbb{Q}) = \sum_p m_{n-(r+1-h(p)),p}.$$

**Remark 4.6.** *Regarding a complex arrangement in  $\mathbb{A}_{\mathbb{C}}^n$  as a real arrangement in  $\mathbb{A}_{\mathbb{R}}^{2n}$ , the formula for the Betti numbers of the complement of a complex arrangement follows from the formula for real arrangements.*

**Example:** Consider  $I = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4) \subseteq k[x_1, x_2, x_3, x_4, x_5]$ . We have

$$CC(H_I^3(R)) = T_{X_{(1,1,0,0,1)}}^* X + T_{X_{(0,0,1,1,1)}}^* X.$$

$$CC(H_I^4(R)) = T_{X_{(1,1,1,1,0)}}^* X + 2 T_{X_{(1,1,1,1,1)}}^* X.$$

Therefore

$$\dim_{\mathbb{Q}} \tilde{H}_2(\mathbb{A}_{\mathbb{R}}^5 - \mathcal{A}; \mathbb{Q}) = 2. \quad \dim_{\mathbb{Q}} \tilde{H}_5(\mathbb{A}_{\mathbb{C}}^5 - \mathcal{A}; \mathbb{Q}) = 2.$$

$$\dim_{\mathbb{Q}} \tilde{H}_3(\mathbb{A}_{\mathbb{R}}^5 - \mathcal{A}; \mathbb{Q}) = 3. \quad \dim_{\mathbb{Q}} \tilde{H}_7(\mathbb{A}_{\mathbb{C}}^5 - \mathcal{A}; \mathbb{Q}) = 1.$$

$$\dim_{\mathbb{Q}} \tilde{H}_8(\mathbb{A}_{\mathbb{C}}^5 - \mathcal{A}; \mathbb{Q}) = 2.$$

**4.3. A general framework:  $D$ -modules with variation zero.** Even though it provides a lot of information, the characteristic cycle does not describe completely the structure of the local cohomology modules. This fact is reflected in the work of A. Galligo, M. Granger and Ph. Maisonobe [30], [31] where they gave a description of the category of regular holonomic  $D_R$ -modules with support a normal crossing, e.g. local cohomology modules supported on squarefree monomial ideals, using the Riemann-Hilbert correspondence.

We will start considering the analytic situation where the Riemann-Hilbert correspondence takes place. For simplicity we will just consider the local situation where  $R = \mathbb{C}\{x_1, \dots, x_n\}$  is the ring of holomorphic functions in  $X = \mathbb{C}^n$ . The Riemann-Hilbert correspondence establish an equivalence of categories between the category  $\text{Mod}_{hr}(D_R)$  of regular holonomic  $D_R$ -modules and the category  $\text{Perv}(\mathbb{C}^n)$  of perverse sheaves by means of the solutions functor  $\text{Sol}(-) := \mathbb{R} \text{Hom}_{D_R}(-, R)$ .

Denote by  $\text{Perv}^T(\mathbb{C}^n)$  the subcategory of  $\text{Perv}(\mathbb{C}^n)$  of complexes of sheaves of finitely dimensional vector spaces on  $\mathbb{C}^n$  which are perverse relatively to the given stratification of  $T$  [30, I.1], and by  $\text{Mod}_{hr}^T(D_R)$  the full abelian subcategory of the category of regular holonomic  $D_R$ -modules such that their solution complex is an object of  $\text{Perv}^T(\mathbb{C}^n)$ . Then, the above equivalence gives by restriction an equivalence of categories between  $\text{Mod}_{hr}^T(D_R)$  and  $\text{Perv}^T(\mathbb{C}^n)$ .

The category  $\text{Perv}^T(\mathbb{C}^n)$  has been described as a quiver representation in [30]. More precisely, they established an equivalence of categories with the category  $\mathcal{C}^n$  whose objects are families  $\{\mathcal{M}_\alpha\}_{\alpha \in \{0,1\}^n}$  of finitely dimensional  $\mathbb{C}$ -vector spaces, endowed with linear maps

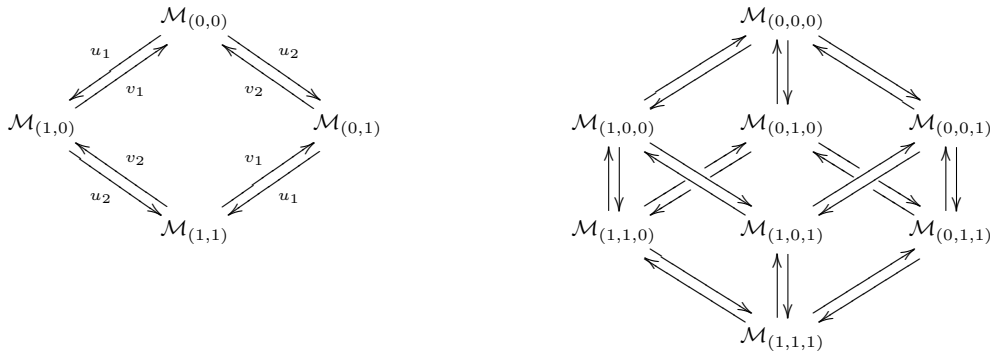
$$\mathcal{M}_\alpha \xrightarrow{u_i} \mathcal{M}_{\alpha+\varepsilon_i} \quad , \quad \mathcal{M}_\alpha \xleftarrow{v_i} \mathcal{M}_{\alpha+\varepsilon_i}$$

for each  $\alpha \in \{0,1\}^n$  such that  $\alpha_i = 0$ . These maps are called canonical (resp. variation) maps, and they are required to satisfy the conditions:

$$u_i u_j = u_j u_i, \quad v_i v_j = v_j v_i, \quad u_i v_j = v_j u_i \quad \text{and} \quad v_i u_i + id \text{ is invertible.}$$

Such an object will be called an  $n$ -hypercube. A morphism between two  $n$ -hypercubes  $\{\mathcal{M}_\alpha\}_\alpha$  and  $\{\mathcal{N}_\alpha\}_\alpha$  is a set of linear maps  $\{f_\alpha : \mathcal{M}_\alpha \rightarrow \mathcal{N}_\alpha\}_\alpha$ , commuting with the canonical and variation maps.

**Example:** The 2-hypercube and the 3-hypercube. We follow the convention that the canonical maps  $u_i$  go downward and the variation maps  $v_i$  go upward:



The construction of the  $n$ -hypercube corresponding to an object  $M$  of  $\text{Mod}_{hr}^T(D_R)$  is explicitly given in [31]. We will skip the details because they are a little bit involved and depend on the solutions of our module  $M$  in some functional spaces. However we

want to point out that the dimension of the  $\mathbb{C}$ -vector spaces  $\mathcal{M}_\alpha$  in the  $n$ -hypercube are determined by the characteristic cycle of  $M$ . More precisely, if  $CC(M) = \sum m_\alpha T_{X_\alpha}^* \mathbb{C}^n$  is the characteristic cycle of  $M$ , then for all  $\alpha \in \{0, 1\}^n$  one has the equality  $\dim_{\mathbb{C}} \mathcal{M}_\alpha = m_\alpha$ . It follows that the characteristic cycle is not enough to characterize a  $D_R$ -module with monomial support, i.e. support in  $T$ . We also need to describe the morphisms  $u_i$ 's and  $v_i$ 's.

*Problem:* Describe the  $n$ -hypercube associated to a local cohomology module  $H_I^r(R)$  supported on a monomial ideal  $I \subseteq R$ ?

To solve this question one needs to consider objects in the category  $\text{Mod}_{hr}^T(D_R)$  having the following property ([5] and [6]):

**Definition 4.7.** *We say that an object  $M$  of  $\text{Mod}_{hr}^T(D_R)$  has variation zero if the morphisms  $v_i$  in the corresponding  $n$ -hypercube are zero for all  $1 \leq i \leq n$  and all  $\alpha \in \{0, 1\}^n$  with  $\alpha_i = 0$ .*

Modules with variation zero form a full abelian subcategory of  $\text{Mod}_{hr}^T(D_R)$  but is not closed under extensions (see [4]). This category will be denoted  $D_{v=0}^T$  and we will denote the corresponding category of  $n$ -hypercubes as  $\mathcal{C}_{v=0}^n$ . We have the following situation

$$\begin{array}{ccccc}
 \text{Mod}_{hr}(D_R) & \xrightarrow{RH\text{-corresp.}} & \text{Perv}(\mathbb{C}^n) & & \\
 \uparrow & & \uparrow & & \\
 \text{Mod}_{hr}^T(D_R) & \longrightarrow & \text{Perv}^T(\mathbb{C}^n) & \xrightarrow{GGM} & \mathbb{C}^n \\
 \uparrow & & \uparrow & & \uparrow \\
 D_{v=0}^T & \longrightarrow & \text{Perv}_{v=0}^T(\mathbb{C}^n) & \longrightarrow & \mathcal{C}_{v=0}^n
 \end{array}$$

A nice characterization of these modules is given in [5] in terms of the following particular filtration given by the height.

**Proposition 4.8.** *An object  $M$  of  $\text{Mod}_{hr}^T(D_R)$  has variation zero if and only if there is an increasing filtration  $\{F_j\}_{0 \leq j \leq n}$  of  $M$  by objects of  $\text{Mod}_{hr}^T(D_R)$  and there are integers  $m_\alpha \geq 0$  for  $\alpha \in \{0, 1\}^n$ , such that for all  $0 \leq j \leq n$  one has  $D_R$ -module isomorphisms*

$$F_j / F_{j-1} \cong \bigoplus_{\substack{\alpha \in \{0, 1\}^n \\ |\alpha| = j}} (H_{\mathfrak{p}_\alpha}^j(R))^{\oplus m_\alpha}.$$

It follows from the degeneraion of the Mayer-Vietoris spectral sequence (see corollary 4.4), that local cohomology modules supported on monomial ideals have variation zero.

**Remark 4.9.** *The solutions of a regular holonomic  $D_R$ -modules are Nilsson class functions, i.e. are finite sums*

$$f = \sum_{\beta, m} \varphi_{\beta, m}(\mathbf{x}) (\log \mathbf{x})^m \mathbf{x}^\beta,$$

where  $\varphi_{\beta, m}(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}$ ,  $\beta \in \mathbb{C}^n$  and  $m \in (\mathbb{Z}^+)^n$  (see [12] for details). A Nilsson class function is a solution of a module with variation zero if and only if  $m = \mathbf{0} \in \mathbb{Z}^n$  and  $\beta \in \mathbb{Z}^n$ , i.e.  $f \in R_{x_1 \dots x_n}$ . This means, roughly speaking, that for a module with variation zero its solutions are algebraic.

*It is easy to check out from its presentation*

$$R_{\mathbf{x}^\alpha} \cong \frac{D_R}{D_R(\{x_i \partial_i + 1 \mid \alpha_i = 1\}, \{\partial_j \mid \alpha_j = 0\})}$$

that localizations at monomials are modules with variation zero. Therefore, using the Čech complex, we can also see that local cohomology modules supported on monomial ideals also belong to the category of modules with variation zero.

4.3.1. *Extension problems.* From the characterization of modules with variation zero given in Proposition 4.8 we obtain a set of short exact sequences

$$\begin{aligned} (s_1) : \quad & 0 \rightarrow F_0 \rightarrow F_1 \rightarrow F_1/F_0 \rightarrow 0 \\ (s_2) : \quad & 0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_2/F_1 \rightarrow 0 \\ & \vdots \\ (s_n) : \quad & 0 \rightarrow F_{n-1} \rightarrow M \rightarrow F_n/F_{n-1} \rightarrow 0 \end{aligned}$$

such that  $F_j/F_{j-1} \cong \bigoplus_{|\alpha|=j} H_{\mathfrak{p}_\alpha}^j(R)^{m_\alpha}$ . It follows from the additivity of the characteristic cycle that

$$CC(M) = \sum m_\alpha T_{X_\alpha}^* X$$

The extension classes of this short exact sequences determine the structure of this module with variation zero so it is not enough considering the characteristic cycle. Recall that each extension class of the sequence  $(s_j)$  defines an element in  $\text{Ext}_{D_R}^1(F_j/F_{j-1}, F_{j-1})$ .

**Example:** Assume that we have a module with variation zero  $M$  in a short exact sequence like

$$0 \rightarrow H_{(x_1, x_2)}^2(R) \rightarrow M \rightarrow H_{(x_1, x_2, x_3)}^3(R) \rightarrow 0$$

Then, its characteristic cycle is  $CC(M) = T_{X_{(1,1,0)}}^* X + T_{X_{(1,1,1)}}^* X$  but the short exact sequence may be split or not. In the first case  $M \cong H_{(x_1, x_2)}^2(R) \oplus H_{(x_1, x_2, x_3)}^3(R)$  but in the second case we have an injective module  $E_{(1,1,0)}$  so is not isomorphic to the sum of local cohomology modules.

To determine a module with variation zero  $M$  we have to solve all the extensions problems associated to the corresponding filtration. On the other hand, to determine the corresponding  $n$ -hypercube we have to describe the linear maps  $u_i$ 's. In section 5 we will make this correspondence more precise with the help of the  $\mathbb{Z}^n$ -graded structure of  $M$  but, for the moment, the reader should notice the following correspondence:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & F_{-1} & \longrightarrow & F_0 & \longrightarrow & F_0/F_{-1} \longrightarrow 0 \\
 \\ 
 0 & \longrightarrow & F_0 & \longrightarrow & F_1 & \longrightarrow & F_1/F_0 \longrightarrow 0 \\
 \\ 
 0 & \longrightarrow & F_1 & \longrightarrow & F_2 & \longrightarrow & F_2/F_1 \longrightarrow 0 \\
 \\ 
 0 & \longrightarrow & F_2 & \longrightarrow & M & \longrightarrow & F_3/F_2 \longrightarrow 0
 \end{array}$$

**4.3.2. Modules with variation zero: The algebraic case.** In principle we are only working in the analytic case but, since the solutions of a module with variation zero are algebraic, the equivalence

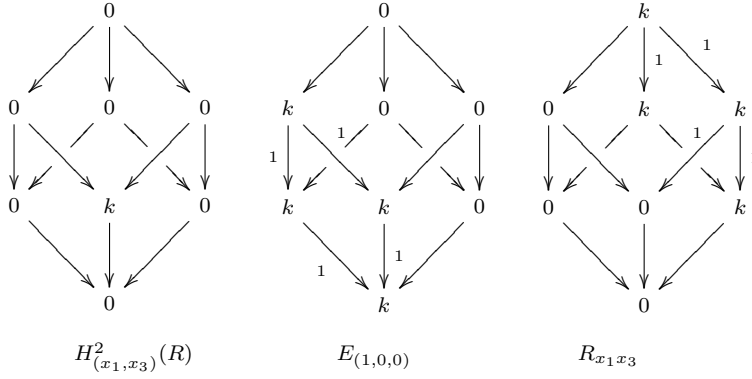
$$D_{v=0}^T \longrightarrow \mathcal{C}_{v=0}^n$$

between the modules with variation zero and the corresponding  $n$ -hypercubes can be extended to the algebraic case over any field of characteristic zero (see [6]). From now on, we will consider the polynomial ring  $R = k[x_1, \dots, x_n]$  over a field of characteristic zero, or even the formal power series ring  $R = k[[x_1, \dots, x_n]]$ .

A very straightforward computation show us that in  $D_{v=0}^T$  we have the following objects,  $\forall \alpha \in \{0, 1\}^n$ :

- **Simple:**  $H_{\mathfrak{p}_\alpha}^{|\alpha|}(R) \cong \frac{R[\frac{1}{\mathbf{x}^\alpha}]}{\sum_{\alpha_i=1} R[\frac{1}{\mathbf{x}^{\alpha-\varepsilon_i}}]} \cong \frac{D_R}{D_R(\{x_i \mid \alpha_i=1\}, \{\partial_j \mid \alpha_j=0\})}.$
- **Injective:**  $E_\alpha := \frac{R[\frac{1}{\mathbf{x}^\alpha}]}{\sum_{\alpha_i=1} R[\frac{1}{\mathbf{x}^{\mathbf{1}-\varepsilon_i}}]} \cong \frac{D_R}{D_R(\{x_i \mid \alpha_i=1\}, \{x_j \partial_j + 1 \mid \alpha_j=0\})}.$
- **Projective:**  $R_{x^\alpha} \cong \frac{D_R}{D_R(\{x_i \partial_i + 1 \mid \alpha_i=1\}, \{\partial_j \mid \alpha_j=0\})}.$

**Example:** The 3-hypercube of a simple, injective and projective module with variation zero is as follows:



The contravariant exact functor  $D_{v=0}^T \longrightarrow \mathcal{C}_{v=0}^n$  can be described explicitly. Namely, given a module with variation zero  $M$ , its corresponding  $n$ -hypercube  $\mathcal{M}$  is constructed as follows:

- i) The vertices of the  $n$ -hypercube are the  $k$ -vector spaces  $\mathcal{M}_\alpha := \text{Hom}_{D_R}(M, E_\alpha)$ .
- ii) The linear maps  $u_i$  are induced by the epimorphisms  $\pi_i : E_\alpha \rightarrow E_{\alpha+\varepsilon_i}$ .

*The positive characteristic case:* Even though we do not have an analogue to the results of [30, 31] in positive characteristic, we can define a category of modules with variation zero and their  $n$ -hypercubes in positive characteristic. In this case one has to define modules with variation zero via the characterization given by the existence of an increasing filtration  $\{F_j\}_{0 \leq j \leq n}$  of submodules of  $M$  such that

$$F_j/F_{j-1} \simeq \bigoplus_{|\alpha|=j} (H_{\mathfrak{p}_\alpha}^{|\alpha|}(R))^{m_\alpha},$$

for some integers  $m_\alpha \geq 0$ ,  $\alpha \in \{0,1\}^n$ . Finally we point out that, using the same arguments as in [5, Lemma 4.4], the  $n$ -hypercube  $\mathcal{M}$  associated to a module with variation zero  $M$  should be constructed using the following variant in terms of  $\mathbb{Z}^n$ -graded morphisms

- i) The vertices of the  $n$ -hypercube are the  $k$ -vector spaces  $\mathcal{M}_\alpha := {}^* \text{Hom}_R(M, E_\alpha)$ .
- ii) The linear maps  $u_i$  are induced by the natural epimorphisms  $\pi_i : E_\alpha \rightarrow E_{\alpha+\varepsilon_i}$ .

## 5. BUILDING A DICTIONARY

The connection between commutative algebra and combinatorics

First of all we will see that the  $D$ -module approach and the  $\mathbb{Z}^n$ -graded approach to the study of local cohomology modules are equivalent

We have already seen that the  $D$ -module and the  $\mathbb{Z}^n$ -graded structure of the local cohomology modules are equivalent. Now we want to make more precise the following equivalences



Local cohomology $H_I^r(R)$		Free resolution $\mathbb{L}_\bullet(I^\vee)$
<i>D-module structure</i>	$\mathbb{Z}^n$ -graded structure	
· Multiplicities CC	· $\mathbb{Z}^n$ -graded pieces	· $\mathbb{Z}^n$ -graded Betti numbers
· $n$ -hypercubes	· Morphisms between pieces	· Linear strands

To describe the  $D$ -module structure of local cohomology modules we introduced the category  $D_{v=0}^T$  of modules with variation zero or equivalently, the category of  $n$ -hypercubes with variation zero  $\mathcal{C}_{v=0}^n$ . On the other hand, to study its  $\mathbb{Z}^n$ -graded structure we used the category **Str** of straight modules.

**5.1.  $D$ -module structure vs.  $\mathbb{Z}^n$ -graded structure.** At first sight we can find the analogies between the categories **Str**,  $D_{v=0}^T$ ,  $\mathcal{C}_{v=0}^n$  of straight modules, modules with variation zero and  $n$ -hypercubes respectively, since we characterized their objects as follows:

- **Straight modules:** A straight module is characterized by the  $\mathbb{Z}^n$ -graded pieces  $M_{-\alpha}$ ,  $\alpha \in \{0, 1\}^n$  and the multiplication by  $x_i$  maps  $x_i : M_{-\alpha} \rightarrow M_{-\alpha + \varepsilon_i}$ .

Equivalently, such a module comes with a filtration of  $\mathbb{Z}^n$ -graded submodules  $\{F_j\}_{0 \leq j \leq n}$  such that

$$F_j/F_{j-1} \cong \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ |\alpha|=j}} (H_{\mathfrak{p}_\alpha}^j(\omega_R))^{\oplus m_\alpha}.$$

Thus, in order to describe this module we have to solve the  $\mathbb{Z}^n$ -graded extension problems

$$(s_j) : \quad 0 \rightarrow F_{j-1} \rightarrow F_j \rightarrow F_j/F_{j-1} \rightarrow 0$$

- **Modules with variation zero:** A module with variation zero is characterized by a filtration of  $D_R$ -modules  $\{F_j\}_{0 \leq j \leq n}$  such that

$$F_j/F_{j-1} \cong \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ |\alpha|=j}} (H_{\mathfrak{p}_\alpha}^j(R))^{\oplus m_\alpha}.$$

Then, to describe this module we have to solve the  $D_R$ -modules extension problems

$$(s_j) : \quad 0 \rightarrow F_{j-1} \rightarrow F_j \rightarrow F_j/F_{j-1} \rightarrow 0$$

- **$n$ -hypercubes:** A  $n$ -hypercube  $\mathcal{M}$  is characterized by the pieces  $\mathcal{M}_\alpha$ ,  $\alpha \in \{0, 1\}^n$  and the canonical maps  $u_i : \mathcal{M}_\alpha \rightarrow \mathcal{M}_{\alpha + \varepsilon_i}$

K. Yanagawa already proved in [87] that straight modules are  $D_R$ -modules so it will not come as a surprise that these modules, modulo a shift by  $\mathbf{1}$ , have in fact variation zero. Recall that the local cohomology of the canonical module  $H_I^r(\omega_R) = H_I^r(R(-\mathbf{1})) = H_I^r(R)(-\mathbf{1})$  supported on a monomial ideal  $I$  is straight but we only want to deal with  $H_I^r(R)$  in the category of modules with variation zero.

Let  $\varepsilon - \mathbf{Str}$  be the category of  $\mathbb{Z}^n$ -graded modules  $M$  such that  $M(-\mathbf{1})$  is straight. An equivalence of categories  $\varepsilon - \mathbf{Str} \xrightarrow{\sim} D_{v=0}^T$  is established in [5, Theorem 4.3]. More precisely, a  $\varepsilon$ -straight module when viewed as a  $D_R$ -module has variation zero.

It is also proved in [5, Lemma 4.4] that given  $\varepsilon$ -straight modules  $M$  and  $N$  then we have functorial isomorphisms

$${}^*\mathrm{Ext}_R^i(M, N) \cong \mathrm{Ext}_{D_{v=0}^T}^i(M, N).$$

for all  $i \geq 0$ . Therefore, the extension problems we had to solve to describe our module are equivalent in both categories. The following proposition is also proved in [5]

**Proposition 5.1.** *The extension class  $(s_j)$  is uniquely determined by the  $k$ -linear maps  $\cdot x_i : M_{-\alpha} \rightarrow M_{-\alpha+\varepsilon_i}$  where  $|\alpha| = j$  and  $\alpha_i = 1$ .*

Finally we will make the last remaining equivalence more precise

$$\varepsilon - \mathbf{Str} \xrightarrow{\sim} D_{v=0}^T \xrightarrow{\sim} \mathcal{C}_{v=0}^n$$

Let  $M \in \varepsilon - \mathbf{Str}$  be a  $\varepsilon$ -straight module. The vertices and linear maps of corresponding  $n$ -hypercube  $\mathcal{M} \in \mathcal{C}_{v=0}^n$  can be described from the  $\mathbb{Z}^n$ -graded pieces of  $M$ . Let  $(M_{-\alpha})^*$  be the dual of the  $k$ -vector space defined by the piece of  $M$  of degree  $-\alpha$ ,  $\alpha \in \{0, 1\}^n$ . Then, there are isomorphisms

$$\mathcal{M}_\alpha \cong (M_{-\alpha})^*$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_\alpha & \xrightarrow{u_{\alpha,i}} & \mathcal{M}_{\alpha+\varepsilon_i} \\ \cong \uparrow & & \cong \uparrow \\ (M_{-\alpha})^* & \xrightarrow{(x_i)^*} & (M_{-\alpha-\varepsilon_i})^* \end{array}$$

where  $(x_i)^*$  is the dual of the multiplication by  $x_i$ .

From now on we will loosely use the term *pieces of a module*  $M$  meaning the pieces of the  $n$ -hypercube associated to  $M$  but, if the reader is more comfortable with the  $\mathbb{Z}^n$ -graded point of view, one may also consider the  $\mathbb{Z}^n$ -graded pieces of  $M$  (with the appropriate sign).

For the case of local cohomology modules

$$\cdot x_i : H_I^r(R)_{-\alpha} \longrightarrow H_I^r(R)_{-(\alpha-\varepsilon_i)}$$

corresponds to the morphism

$$\tilde{H}^{r-2}(\Delta_{\mathbf{1}-\alpha}^\vee; k) \longrightarrow \tilde{H}^{r-2}(\Delta_{\mathbf{1}-\alpha-\varepsilon_i}^\vee; k),$$

induced by the inclusion  $\Delta_{\mathbf{1}-\alpha-\varepsilon_i}^\vee \subseteq \Delta_{\mathbf{1}-\alpha}^\vee$ .

$$[H_I^r(R)]_{-\alpha} \cong (\tilde{H}_{r-2}(T_\alpha; \mathbb{C}))^* \cong \tilde{H}^{r-2}(T_\alpha; \mathbb{C}), \quad \alpha \in \{0, 1\}^n,$$

and the multiplication map  $x_i: [H_I^r(R)]_{-\alpha-\varepsilon_i} \rightarrow [H_I^r(R)]_{-\alpha}$  is determined by the following commutative diagram:

$$\begin{array}{ccc} ([H_I^r(R)]_{-\alpha})^* & \xrightarrow{(x_i)^*} & ([H_I^r(R)]_{-\alpha-\varepsilon_i})^* \\ \cong \uparrow & & \uparrow \cong \\ ([H_I^r(R)]_\alpha) & \xrightarrow{u_i} & ([H_I^r(R)]_{\alpha+\varepsilon_i}) \\ \cong \downarrow & & \downarrow \cong \\ (\tilde{H}^{r-2}(T_\alpha; k))^* & \xrightarrow{(\nu_i)^*} & (\tilde{H}^{r-2}(T_{\alpha+\varepsilon_i}; k))^* \end{array}$$

where  $\nu_i$  is induced by the inclusion  $\Delta_{\alpha-\varepsilon_i}^\vee \subseteq \Delta_\alpha^\vee$ .

**Remark 5.2.** *The advantage of the  $D$ -module approach is that it is more likely to be extended to other situations like the case of hyperplane arrangements. We recall that local cohomology modules with support an arrangement of linear subvarieties were already computed in [5] and a quiver representation of  $D_{R|k}$ -modules with support a hyperplane arrangement is given in [49], [50].*

**5.2. Local cohomology vs. free resolutions.** Let  $R = k[x_1, \dots, x_n]$  be the polynomial ring in  $n$  variables over a field  $k$  of any characteristic. Let  $I = \mathfrak{p}_{\alpha_1} \cap \dots \cap \mathfrak{p}_{\alpha_m}$  be the minimal primary decomposition of a squarefree monomial ideal. Its Alexander dual ideal  $I^\vee$  is of the form  $I^\vee = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_m})$ . The aim of this section is to relate the structure of the local cohomology modules  $H_I^r(R)$  to the structure of the minimal free resolution of the ideal  $I^\vee$ .

**5.2.1. Characteristic cycle vs. Betti numbers.** M. Mustařă [66, Cor. 3.1] already proved the following relation between the pieces of the local cohomology modules and the Betti numbers of the Alexander dual ideal

$$\beta_{j,\alpha}(I^\vee) = \dim_k [H_I^{|\alpha|-j}(R)]_\alpha$$

so the pieces of  $H_I^r(R)$  for a fixed  $r$  describe the modules and the Betti numbers of the  $r$ -linear strand of  $I^\vee$ . Recall that  $\dim_k [H_I^r(R)]_\alpha = m_{n-r,\alpha}$ , where  $CC(H_I^r(R)) = \sum m_{n-r,\alpha} T_{X_\alpha}^* X$  is the characteristic cycle of the local cohomology module. Therefore we have:

**Proposition 5.3.** *Let  $I^\vee \subseteq R$  be Alexander dual ideal of a squarefree monomial ideal  $I \subseteq R$ . Then we have:*

$$\beta_{j,\alpha}(I^\vee) = m_{n-|\alpha|+j,\alpha}(R/I).$$

The methods we used in Section 4.1 to compute the multiplicities of the characteristic cycle of local cohomology modules can be interpreted as follows:

· *Mayer-Vietoris process*: The initial pieces that we need to start the process are the local cohomology modules supported on sums of face ideals in the minimal primary decomposition  $I = \mathfrak{p}_{\alpha_1} \cap \cdots \cap \mathfrak{p}_{\alpha_m}$ . These sums are again face ideals and it is not difficult to check out that their Alexander duals are the least common multiples of the Alexander dual of each face ideal in the sum, i.e.

$$(\mathfrak{p}_{\alpha_{\ell_1}} + \cdots + \mathfrak{p}_{\alpha_{\ell_r}})^\vee = \text{lcm}(\mathbf{x}^{\alpha_{\ell_1}}, \dots, \mathbf{x}^{\alpha_{\ell_r}})$$

Thus, the information encoded by the initial pieces allow us to construct the Taylor resolution of the ideal  $I^\vee$ . The Mayer-Vietoris process can be understood as the process to pass from the Taylor resolution to a minimal free resolution of the ideal  $I^\vee$ .

· *Mayer-Vietoris spectral sequence*: The  $E_1$ -page of the Mayer-Vietoris spectral sequence also encodes the information needed to construct the Taylor resolution of the ideal  $I^\vee$ . If we take a close look to the  $E_2$ -page we will see that the poset  $P(\mathcal{A})$  associated to the ideal  $I$  is nothing but the lcm-lattice of the Alexander dual ideal  $I^\vee$ . Therefore one recovers the formula for the Betti numbers given in [33].

**Example:** Let  $I = (x_1, x_2) \cap (x_1, x_3) \cap (x_2, x_3) \subseteq k[x_1, x_2, x_3]$ . The initial pieces that we use in the Mayer-Vietoris process that allow us to compute the characteristic cycle of the local cohomology modules  $H_I^r(R)$  are

$$\begin{array}{lll} H_{(x_1, x_2)}^2(R), & H_{(x_1, x_2, x_3)}^3(R), & H_{(x_1, x_2, x_3)}^3(R) \\ H_{(x_1, x_3)}^2(R), & H_{(x_1, x_2, x_3)}^3(R), & \\ H_{(x_2, x_3)}^2(R), & H_{(x_1, x_2, x_3)}^3(R), & \end{array}$$

corresponding to the sums of 1, 2 and 3 ideals in the minimal primary decomposition of  $I$ . Equivalently, these are the modules that appear in the  $E_1$ -page of the Mayer-Vietoris spectral sequence. The information given by these modules is equivalent to the information needed to describe the Taylor resolution of the Alexander dual ideal  $I^\vee = (x_1x_2, x_1x_3, x_2x_3)$ .

$$\begin{array}{ccccccc} & & R(-1, -1, -1) & & R(-1, -1, 0) & & \\ & & \oplus & & \oplus & & \\ 0 \longrightarrow & R(-1, -1, -1) & \longrightarrow & R(-1, -1, -1) & \longrightarrow & R(-1, 0, -1) & \longrightarrow I^\vee \longrightarrow 0 \\ & & \oplus & & \oplus & & \\ & & R(-1, -1, -1) & & R(0, -1, -1) & & \end{array}$$

If we apply our Mayer-Vietoris process or compute the  $E_2$ -page of the Mayer-Vietoris spectral sequence we obtain the characteristic cycle

$$CC(H_I^2(R)) = T_{X_{(1,1,0)}}^* X + T_{X_{(1,0,1)}}^* X + T_{X_{(0,1,1)}}^* X + 2 T_{X_{(1,1,1)}}^* X$$

that corresponds to the minimal free resolution

$$\begin{array}{ccccccc} & & & R(-1, -1, 0) & & & \\ & & & \oplus & & & \\ 0 & \longrightarrow & R(-1, -1, -1) & \longrightarrow & R(-1, 0, -1) & \longrightarrow & I^\vee \longrightarrow 0 \\ & & \oplus & & \oplus & & \\ & & R(-1, -1, -1) & & R(0, -1, -1) & & \end{array}$$

If  $R/I$  is Cohen-Macaulay then there is only one non vanishing local cohomology module so we can recover the following fundamental result of J. A. Eagon and V. Reiner [21].

**Corollary 5.4.** *Let  $I^\vee \subseteq R$  be the Alexander dual ideal of a squarefree monomial ideal  $I \subseteq R$ . Then,  $R/I$  is Cohen-Macaulay if and only if  $I^\vee$  has a linear free resolution.*

A generalization of this result expressed in terms of the projective dimension of  $R/I$  and the Castelnuovo-Mumford regularity of  $I^\vee$  is given by N. Terai in [81]. We can also give a different approach by using the previous results.

**Corollary 5.5.** *Let  $I^\vee \subseteq R$  be the Alexander dual ideal of a squarefree monomial ideal  $I \subseteq R$ . Then we have:*

$$\text{pd}(R/I) = \text{reg}(I^\vee).$$

PROOF: By using Proposition 5.3 we have:

$$\text{reg}(I^\vee) := \max \{ |\alpha| - j \mid \beta_{j,\alpha}(I^\vee) \neq 0 \} = \max \{ |\alpha| - j \mid m_{n-|\alpha|+j,\alpha}(R/I) \neq 0 \}.$$

Then, by [1, Corollary 3.13] we get the desired result since:

$$\text{pd}(R/I) = \text{cd}(R, I) = \max \{ |\alpha| - j \mid m_{n-|\alpha|+j,\alpha}(R/I) \neq 0 \},$$

where the first assertion comes from [53].

*Mayer-Vietoris splittings:* Splittings of a monomial ideal have a long tradition in the study of free resolutions (see [25], [26], [40]). One looks for a decomposition  $I = J + K$  of our ideal satisfying the following formula for the  $\mathbb{Z}^n$ -graded Betti numbers

$$\beta_{i,\alpha}(I) = \beta_{i,\alpha}(J) + \beta_{i,\alpha}(K) + \beta_{i-1,\alpha}(J \cap K)$$

C. Francisco, H. T. Hà and A. Van Tuyl coined the term Betti splitting in [28] to tackle this formula. Using our approach it is easy to check out that the condition of being a Betti splitting is nothing but the splitting of the Mayer-Vietoris exact sequence for the local cohomology modules of its Alexander dual  $I^\vee = J^\vee \cap K^\vee$

$$\cdots \longrightarrow H_{J^\vee + K^\vee}^r(R) \longrightarrow H_{J^\vee}^r(R) \oplus H_{K^\vee}^r(R) \longrightarrow H_{I^\vee}^r(R) \longrightarrow H_{J^\vee + K^\vee}^{r+1}(R) \longrightarrow \cdots$$

**Definition 5.6.** We say that a squarefree monomial ideal  $I^\vee = J^\vee \cap K^\vee$  is  $r$ -MV-splittable if the corresponding Mayer-Vietoris exact sequence splits at level  $r$ , i.e. we have a short exact sequence

$$0 \longrightarrow H_{J^\vee}^r(R) \oplus H_{K^\vee}^r(R) \longrightarrow H_{I^\vee}^r(R) \longrightarrow H_{J^\vee + K^\vee}^{r+1}(R) \longrightarrow 0$$

A squarefree monomial ideal  $I^\vee$  is MV-splittable if it is  $r$ -MV-splittable for all  $r$ .

The following result comes easily from Proposition 5.2

**Proposition 5.7.** A squarefree monomial ideal  $I$  is Betti-splittable with splitting  $I = J + K$  if and only if  $I^\vee = J^\vee \cap K^\vee$  is MV-splittable. In this case, for all  $i \geq 0$ ,  $\alpha \in \{0, 1\}^n$

$$\beta_{i,\alpha}(I) = \beta_{i,\alpha}(J) + \beta_{i,\alpha}(K) + \beta_{i-1,\alpha}(J \cap K)$$

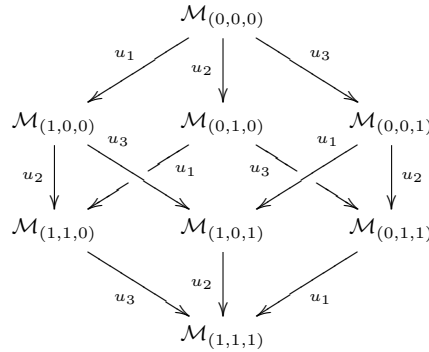
The condition of  $I^\vee$  being  $r$ -MV-splittable is equivalent to have the same formula for the Betti numbers in the  $r$ -linear strand.

5.2.2.  $n$ -hypercubes vs. linear strands. First, notice that, giving the appropriate sign to the canonical maps of the hypercube  $\mathcal{M} = \{\mathcal{M}_\alpha\}_\alpha$  associated to a module with variation zero  $M$ , we can construct the following complex of  $k$ -vector spaces:

$$\mathcal{M}^\bullet : 0 \longleftarrow \mathcal{M}_1 \xleftarrow{u_0} \bigoplus_{|\alpha|=n-1} \mathcal{M}_\alpha \xleftarrow{u_1} \cdots \xleftarrow{u_{p-1}} \bigoplus_{|\alpha|=n-p} \mathcal{M}_\alpha \xleftarrow{u_p} \cdots \xleftarrow{u_{n-1}} \mathcal{M}_0 \longleftarrow 0$$

where the map between summands  $\mathcal{M}_\alpha \longrightarrow \mathcal{M}_{\alpha+\varepsilon_i}$  is  $\text{sign}(i, \alpha + \varepsilon_i)$  times the canonical map  $u_i : \mathcal{M}_\alpha \longrightarrow \mathcal{M}_{\alpha+\varepsilon_i}$ .

**Example:** 3-hypercube and its associated complex



$$0 \longleftarrow \mathcal{M}_{(1,1,1)} \xleftarrow{(u_3, u_2, u_1)} \begin{array}{c} \mathcal{M}_{(1,1,0)} \\ \oplus \\ \mathcal{M}_{(1,0,1)} \\ \oplus \\ \mathcal{M}_{(0,1,1)} \end{array} \xleftarrow{\begin{pmatrix} -u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \\ 0 & u_3 & u_2 \end{pmatrix}} \begin{array}{c} \mathcal{M}_{(1,0,0)} \\ \oplus \\ \mathcal{M}_{(0,1,0)} \\ \oplus \\ \mathcal{M}_{(0,0,1)} \end{array} \xleftarrow{\begin{pmatrix} -u_1 \\ u_2 \\ -u_3 \end{pmatrix}} \mathcal{M}_{(0,0,0)} \longleftarrow 0$$

Our main result in this section is that the matrices in the complex of  $k$ -vector spaces associated to the  $n$ -hypercube of a fixed local cohomology module  $H_I^r(R)$  are the transpose of the monomial matrices of the  $r$ -linear strand<sup>6</sup> of the Alexander dual ideal  $I^\vee$ . To prove the following proposition one has to put together some results scattered in the work of K. Yanagawa [86, 87].

**Proposition 5.8.** *Let  $\mathcal{M} = \{[H_I^r(R)]_\alpha\}_{\alpha \in \{0,1\}^n}$  be the  $n$ -hypercube of a fixed local cohomology module  $H_I^r(R)$  supported on a monomial ideal  $I \subseteq R = k[x_1, \dots, x_n]$ . Then,  $\mathcal{M}^\bullet$  is the complex of  $k$ -vector spaces whose matrices are the transpose of the monomial matrices of the  $r$ -linear strand  $\mathbb{L}_{\bullet}^{<r>}(I^\vee)$  of the Alexander dual ideal of  $I$ .*

PROOF: Given an squarefree module  $M$ , K. Yanagawa constructed in [86] a chain complex  $\mathbb{F}_\bullet(M)$  of free  $R$ -modules as follows:

$$\mathbb{F}_\bullet(M) : 0 \longrightarrow [M]_1 \otimes_k R \xrightarrow{d_0} \cdots \xrightarrow{d_{p-1}} \bigoplus_{|\alpha|=n-p} [M]_\alpha \otimes_k R \xrightarrow{d_p} \cdots \xrightarrow{d_{n-1}} [M]_0 \otimes_k R \longrightarrow 0$$

where the map between summands  $[M]_{\alpha+\varepsilon_i} \otimes_k R \longrightarrow [M]_\alpha \otimes_k R$  sends  $y \otimes 1 \in [M]_{\alpha+\varepsilon_i} \otimes_k R$  to  $\text{sign}(i, \alpha + \varepsilon_i) (x_i y \otimes x_i)$ . For the particular case of  $M = \text{Ext}_R^r(R/I, R(-\mathbf{1}))$  he proved an isomorphism (after an appropriate shifting) between  $\mathbb{F}_\bullet(M)$  and the  $r$ -linear strand  $\mathbb{L}_{\bullet}^{<r>}(I^\vee)$  of the Alexander dual ideal  $I^\vee$  of  $I$  (see [86, Thm 4.1]).

We have an equivalence of categories between squarefree modules and straight modules [87, Prop. 2.8], thus one may also construct the chain complex  $\mathbb{F}_\bullet(M)$  for any straight module  $M$ . The squarefree module  $\text{Ext}_R^r(R/I, R(-\mathbf{1}))$  corresponds to the local cohomology modules  $H_I^r(R)(-\mathbf{1})$  so there is an isomorphism between  $\mathbb{F}_\bullet(H_I^r(R)(-\mathbf{1}))$  and the  $r$ -linear strand  $\mathbb{L}_{\bullet}^{<r>}(I^\vee)$  after an appropriate shifting. Taking a close look to the construction of  $\mathbb{F}_\bullet(M)$  one may check that the scalar entries in the corresponding monomial matrices are obtained by transposing the scalar entries in the one associated to the hypercube of  $H_I^r(R)$  with the appropriate shift. More precisely, if

$$\mathbb{L}_{\bullet}^{<r>}(I^\vee) : 0 \longrightarrow L_{n-r}^{<r>} \longrightarrow \cdots \longrightarrow L_1^{<r>} \longrightarrow L_0^{<r>} \longrightarrow 0,$$

is the  $r$ -linear strand of the Alexander dual ideal  $I^\vee$  then we transpose its monomial matrices to obtain a complex of  $k$ -vector spaces indexed as follows:

$$\mathbb{F}_{\bullet}^{<r>}(I^\vee)^* : 0 \longleftarrow K_0^{<r>} \longleftarrow \cdots \longleftarrow K_{n-r-1}^{<r>} \longleftarrow K_{n-r}^{<r>} \longleftarrow 0$$

### Lecture 3: Bass numbers of local cohomology modules

Throughout this section we will consider either the polynomial ring  $R = [x_1, \dots, x_n]$  or the formal power series ring  $R = [[x_1, \dots, x_n]]$  over a field  $k$ . Given a squarefree monomial

<sup>6</sup>In the language of [70] we would say that the  $n$ -hypercube has the same information as the frame of the  $r$ -linear strand

ideal  $I \subseteq R$  we are going to study the Bass numbers of the local cohomology modules  $H_I^r(R)$ . These modules are modules with variation zero so we are going to work in this framework and study Bass numbers of such modules.

## 6. BASS NUMBERS OF MODULES WITH VARIATION ZERO

Let  $M \in D_{v=0}^T$  be a module with variation zero. The aim of this section is to compute the pieces of the local cohomology module  $H_{\mathfrak{p}_\alpha}^p(M)$ , for any given homogeneous prime ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^n$ . This module also belongs to  $D_{v=0}^T$  so we want to compute the pieces of the corresponding  $n$ -hypercube  $\{[H_{\mathfrak{p}_\alpha}^p(M)]_\beta\}_{\beta \in \{0, 1\}^n} \in \mathcal{C}_{v=0}^n$ . Among these pieces we find the Bass numbers of  $M$  (see [4, Prop. 4.1]). Namely, we have

$$\mu_p(\mathfrak{p}_\alpha, M) = \dim_k [H_{\mathfrak{p}_\alpha}^p(M)]_\alpha$$

Bass numbers have a good behavior with respect to localization so we can always assume that  $\mathfrak{p}_\alpha = \mathfrak{m}$  is the maximal ideal and  $\mu_p(\mathfrak{m}, M) = \dim_k [H_{\mathfrak{m}}^p(M)]_1$ .

**Remark 6.1.** Let  $\mathcal{M} \in \mathcal{C}_{v=0}^n$  be an  $n$ -hypercube. The restriction of  $\mathcal{M}$  to a face ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^n$  is the  $|\alpha|$ -hypercube  $\mathcal{M}_{\leq \alpha} := \{\mathcal{M}_\beta\}_{\beta \leq \alpha} \in \mathcal{C}_{v=0}^{|\alpha|}$  (see [4, Prop. 3.1]). This gives a functor that in some cases plays the role of the localization functor. In particular, to compute the Bass numbers with respect to  $\mathfrak{p}_\alpha$  of a module with variation zero  $M$  we only have to consider the corresponding  $|\alpha|$ -hypercube  $\mathcal{M}_{\leq \alpha}$  so we may assume that  $\mathfrak{p}_\alpha$  is the maximal ideal.

Later on we will specialize to the case of  $M$  being a local cohomology module  $H_I^r(R)$ . In particular, we will give a different approach to the computation of the Bass numbers of these modules given by K. Yanagawa in [87] or the algorithmic computation given by the author in [1], [2].

**6.1. Computing Bass numbers.** The degree **1** part of the hypercube corresponding to the local cohomology module  $H_{\mathfrak{m}}^p(M)$  is the  $p$ -th homology of the complex of  $k$ -vector spaces

$$[\check{C}_{\mathfrak{m}}(M)]_1^\bullet : 0 \longleftarrow [M]_1 \xleftarrow{\overline{d_0}} \bigoplus_{|\alpha|=1} [M_{\mathbf{x}^\alpha}]_1 \xleftarrow{\overline{d_1}} \cdots \xleftarrow{\overline{d_{p-1}}} \bigoplus_{|\alpha|=p} [M_{\mathbf{x}^\alpha}]_1 \xleftarrow{\overline{d_p}} \cdots \xleftarrow{\overline{d_{n-1}}} [M_{\mathbf{x}^1}]_1 \longleftarrow 0$$

that we obtain applying the exact functor<sup>7</sup>  $\mathrm{Hom}_{D_R}(\cdot, E_1)$  to the Čech complex

$$\check{C}_{\mathfrak{m}}^\bullet(M) : 0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{|\alpha|=1} M_{\mathbf{x}^\alpha} \xrightarrow{d_1} \cdots \xrightarrow{d_{p-1}} \bigoplus_{|\alpha|=p} M_{\mathbf{x}^\alpha} \xrightarrow{d_p} \cdots \xrightarrow{d_{n-1}} M_{\mathbf{x}^1} \longrightarrow 0,$$

<sup>7</sup>In positive characteristic we apply the functor  ${}^*\mathrm{Hom}_R(\cdot, E_1)$



On the other hand, associated to the hypercube  $\mathcal{M} = \{\mathcal{M}_\alpha\}_\alpha$  we constructed a complex of  $k$ -vector spaces:

$$\mathcal{M}^\bullet : 0 \longleftarrow \mathcal{M}_1 \xleftarrow{u_0} \bigoplus_{|\alpha|=n-1} \mathcal{M}_\alpha \xleftarrow{u_1} \cdots \xleftarrow{u_{p-1}} \bigoplus_{|\alpha|=n-p} \mathcal{M}_\alpha \xleftarrow{u_p} \cdots \xleftarrow{u_{n-1}} \mathcal{M}_0 \longleftarrow 0$$

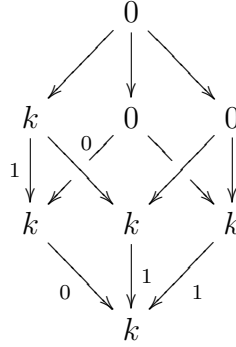
**Proposition 6.2.** *Let  $M \in D_{v=0}^T$  be a module with variation zero and  $\mathcal{M}^\bullet$  its corresponding complex associated to the  $n$ -hypercube. Then, there is an isomorphism of complexes  $\mathcal{M}^\bullet \cong [\check{C}_{\mathfrak{m}}(M)]_1^\bullet$ . In particular  $[H_{\mathfrak{m}}^p(M)]_1 \cong H_p(\mathcal{M}^\bullet)$ .*

Therefore we have the following characterization of Bass numbers:

**Corollary 6.3.** *Let  $M \in D_{v=0}^T$  be a module with variation zero and  $\mathcal{M}^\bullet$  its corresponding complex associated to the  $n$ -hypercube. Then*

$$\mu_p(\mathfrak{m}, M) = \dim_k H_p(\mathcal{M}^\bullet)$$

**Example:** Let  $R = k[x_1, x_2, x_3]$ . Consider the 3-hypercube



of a random module  $M \in D_{v=0}^T$ . It is not difficult to check out that in fact

$$M \cong (H_{(x_1)}^1(R))_{x_2} \oplus H_{(x_1 x_2, x_3)}^2(R)$$

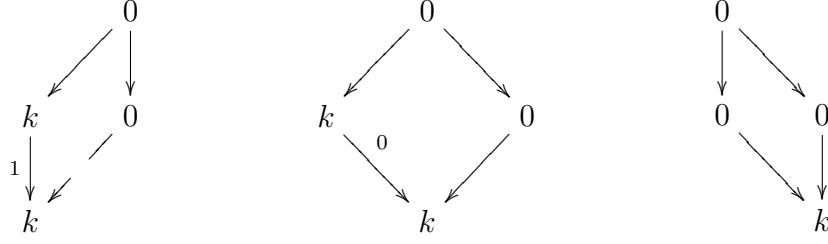
but this is not important for our purposes. We are going to compute the Bass numbers with respect to any homogeneous prime ideal  $\mathfrak{p}_\alpha$ ,  $\alpha \in \{0, 1\}^3$ .

- For  $\alpha = (1, 1, 1)$ , the complex  $\mathcal{M}^\bullet$  associated to the 3-hypercube is:

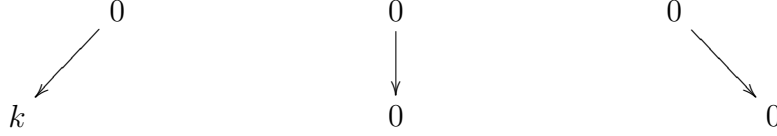
$$0 \longleftarrow k \xleftarrow{(0,1,1)} k^3 \xleftarrow{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}} k \longleftarrow 0 \longleftarrow 0$$

Therefore  $\mu_1(\mathfrak{m}, M) = \dim_k H_1(\mathcal{M}^\bullet) = 1$  and  $\mu_p(\mathfrak{m}, M) = 0$  for all  $p \neq 1$ .

• The restriction of  $\mathcal{M}$  to the face ideals  $\mathfrak{p}_\alpha$  for  $\alpha = (1, 1, 0), (1, 0, 1)$  and  $(0, 1, 1)$  are respectively



• The restriction of  $\mathcal{M}$  to the face ideals  $\mathfrak{p}_\alpha$  for  $\alpha = (1, 0, 0), (0, 1, 0)$  and  $(0, 0, 1)$  are



After computing the homology of the associated complexes of  $k$ -vector spaces, the Bass numbers of  $M$  are:

$\mathfrak{p}_\alpha$	$\mu_0$	$\mu_1$	$\mu_2$
$(x_1)$	1	-	-
$(x_2)$	-	-	-
$(x_3)$	-	-	-
$(x_1, x_2)$	-	-	-
$(x_1, x_3)$	1	1	-
$(x_2, x_3)$	1	-	-
$(x_1, x_2, x_3)$	-	1	-

6.1.1. *Extracting some information.* Once we compute Bass numbers of modules with variation zero we can deal with the following questions:

- Annihilation of Bass numbers.
- Injective dimension.
- Description of the small support of local cohomology modules.
- Associated primes of local cohomology modules.

**Example:** Consider the module with variation zero in the previous example. Notice that we have  $\text{Supp}_R M = V(x_1) \cup V(x_2, x_3)$ , so  $\dim_R \text{Supp}_R M = 2$ . In this case, the inclusion  $\text{supp}_R M \subset \text{Supp}_R M$  is strict since  $(x_1, x_2)$  does not belong to the small support.

On the other hand, the  $\mathbb{Z}^3$ -graded injective dimension is  ${}^*\text{id}_R M = 1$ . Finally, the set of associated primes, i.e. those prime ideals having 0-th Bass number different from zero are

$$\text{Ass}_R M = \{(x_1), (x_1, x_3), (x_2, x_3)\}$$

In particular  $(x_1, x_3)$  is an associated prime that is not minimal in the support of  $M$ .

**6.2. Computing Lyubeznik numbers.** Now we specialize to the case where our module with variation zero is a local cohomology module  $H_I^r(R)$  supported on a monomial ideal  $I \subseteq R$ . We want to compute

$$\lambda_{p,n-r}(R/I) = \mu_p(\mathfrak{m}, H_I^r(R)) = \dim_k [H_{\mathfrak{m}}^p(H_I^r(R))]_1$$

**Remark 6.4.** *Our approach is different from [87] where K. Yanagawa gave the following formula for Lyubeznik numbers:*

$$\lambda_{p,n-r}(R/I) = \dim_k [\text{Ext}_R^{n-p}(\text{Ext}_R^{n-r}(R/I, R), R)]_0$$

Let  $\mathcal{M} = \{[H_I^r(R)]_{\alpha}\}_{\alpha \in \{0,1\}^n}$  be the  $n$ -hypercube corresponding to  $H_I^r(R)$ . In this case we have a topological description of the pieces and linear maps of the  $n$ -hypercube, e.g. using M. Mustaș's approach [66]. Thus, the complex of  $k$ -vector spaces associated to  $\mathcal{M}$  is:

$$\mathcal{M}^{\bullet} : 0 \longleftarrow \tilde{H}^{r-2}(\Delta_0^{\vee}; k) \xleftarrow{u_0} \cdots \xleftarrow{u_{p-1}} \bigoplus_{|\alpha|=p} \tilde{H}^{r-2}(\Delta_{\alpha}^{\vee}; k) \xleftarrow{u_p} \cdots \xleftarrow{u_{n-1}} \tilde{H}^{r-2}(\Delta_1^{\vee}; k) \longleftarrow 0$$

where the map between summands  $\tilde{H}^{r-2}(\Delta_{\alpha+\varepsilon_i}^{\vee}; k) \longrightarrow \tilde{H}^{r-2}(\Delta_{\alpha}^{\vee}; k)$ , is induced by the inclusion  $\Delta_{\alpha}^{\vee} \subseteq \Delta_{\alpha+\varepsilon_i}^{\vee}$ . In particular, the Lyubeznik numbers of  $R/I$  are

$$\lambda_{p,n-r}(R/I) = \dim_k H_p(\mathcal{M}^{\bullet})$$

At this point one may wonder whether there is a simplicial complex, a regular cell complex, or a CW-complex that supports the complex of  $k$ -vector spaces  $\mathcal{M}^{\bullet}$  so one may get a Hochster-like formula not only for the pieces of the local cohomology modules  $H_I^r(R)$  but for its Bass numbers as well. Unfortunately this is not the case in general. To check this out we will use our dictionary developed in section 5 to make a detour through the theory of free resolutions of monomial ideals. Then we refer to the work of M. Velasco [83] to find examples of free resolutions that are not supported by CW-complexes.

**6.2.1. An interpretation of Lyubeznik numbers.** Let  $\mathcal{M} = \{[H_I^r(R)]_{\alpha}\}_{\alpha \in \{0,1\}^n}$  be the  $n$ -hypercube associated to a fixed local cohomology module  $H_I^r(R)$  supported on a monomial ideal  $I \subseteq R = k[x_1, \dots, x_n]$

Consider the  $r$ -linear strand of the Alexander dual ideal  $I^{\vee}$

$$\mathbb{L}_{\bullet}^{<r>}(I^{\vee}) : 0 \longrightarrow L_{n-r}^{<r>} \longrightarrow \cdots \longrightarrow L_1^{<r>} \longrightarrow L_0^{<r>} \longrightarrow 0$$

Transposing its monomial matrices we obtain a complex of  $k$ -vector spaces

$$\mathbb{F}_{\bullet}^{<r>}(I^{\vee})^* : 0 \longleftarrow K_0^{<r>} \longleftarrow \cdots \longleftarrow K_{n-r-1}^{<r>} \longleftarrow K_{n-r}^{<r>} \longleftarrow 0$$

that is isomorphic to the complex of  $k$ -vector spaces  $\mathcal{M}^\bullet$  associated to the  $n$ -hypercube (see Proposition 5.5). Then we have

**Corollary 6.5.** *Let  $\mathbb{F}_\bullet^{\langle r \rangle}(I^\vee)^*$  be the complex of  $k$ -vector spaces obtained from the  $r$ -linear strand of the minimal free resolution of the Alexander dual ideal  $I^\vee$  transposing its monomial matrices. Then*

$$\lambda_{p,n-r}(R/I) = \dim_k H_p(\mathbb{F}_\bullet^{\langle r \rangle}(I^\vee)^*)$$

It follows that one may think Lyubeznik numbers of a squarefree monomial  $I$  as a measure of the acyclicity of the  $r$ -linear strand of the Alexander dual  $I^\vee$ .

**Remark 6.6.** *As a summary of the dictionary between local cohomology modules and free resolutions we have:*

- The pieces  $[H_I^r(R)]_\alpha$  correspond to the Betti numbers  $\beta_{|\alpha|-r,\alpha}(I^\vee)$
- The  $n$ -hypercube of  $H_I^r(R)$  corresponds to the  $r$ -linear strand  $\mathbb{L}_\bullet^{\leq r}(I^\vee)$

Given a free resolution  $\mathbb{L}_\bullet$  of a finitely generated graded  $R$ -module  $M$ , D. Eisenbud, G. Fløystad and F.O. Schreyer [23] defined its linear part as the complex  $\text{lin}(\mathbb{L}_\bullet)$  obtained by erasing the terms of degree  $\geq 2$  from the matrices of the differential maps. To measure the acyclicity of the linear part, J. Herzog and S. Iyengar [43] introduced the linearity defect of  $M$  as  $\text{ld}_R(M) := \sup\{p \mid H_p(\text{lin}(\mathbb{L}_\bullet))\}$ . Therefore we also have:

- The  $n$ -hypercubes of  $H_I^r(R)$ ,  $\forall r$  correspond to the linear part  $\text{lin}(\mathbb{L}_\bullet(I^\vee))$
- The Lyubeznik table of  $R/I$  can be viewed as a generalization of  $\text{ld}_R(I^\vee)$

**6.2.2. Examples.** It is well-known that Cohen-Macaulay squarefree monomial ideals have a trivial Lyubeznik table

$$\Lambda(R/I) = \begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}$$

because they only have one non-vanishing local cohomology module. Recall that its Alexander dual has a linear resolution (see [21, Thm. 3]) so its acyclic. In general, there are non-Cohen-Macaulay ideals with trivial Lyubeznik table. Some of them are far from having only one local cohomology module different from zero.

**Example:** Consider the ideal in  $k[x_1, \dots, x_9]$ :

$$I = (x_1, x_2) \cap (x_3, x_4) \cap (x_5, x_6) \cap (x_7, x_8) \cap (x_9, x_1) \cap (x_9, x_2) \cap (x_9, x_3) \cap (x_9, x_4) \cap (x_9, x_5) \cap \\ \cap (x_9, x_6) \cap (x_9, x_7) \cap (x_9, x_8)$$

The non-vanishing local cohomology modules are  $H_I^r(R)$ ,  $r = 2, 3, 4, 5$  but the Lyubeznik table is trivial.

Ideals with trivial Lyubeznik table can be characterized by the acyclicity of the linear strands.

**Proposition 6.7.** *Let  $I \subseteq R = k[x_1, \dots, x_n]$  be a squarefree monomial ideal. Then, the following conditions are equivalent:*

- i)  $R/I$  has a trivial Lyubeznik table.
- ii)  $H_{(n-r)-i}(\mathbb{F}_{\bullet}^{<r>}(I^\vee)^*) = 0 \quad \forall i > 0$  if  $r = \text{ht } I$  and  
 $H_{(n-r)-i}(\mathbb{F}_{\bullet}^{<r>}(I^\vee)^*) = 0 \quad \forall i \geq 0$  if  $r \neq \text{ht } I$

Notice that the second condition is close to  $I^\vee$  being *componentwise linear*. This notion was introduced by J. Herzog and T. Hibi in [42] where they also proved that their Alexander dual belong to the class of *sequentially Cohen-Macaulay* ideals given by R. Stanley [80]. On the other hand, K. Yanagawa [86, Prop. 4.9] and T. Römer [75, Thm. 3.2.8] characterized componentwise linear ideals as those having acyclic linear strands in homological degree different from zero. Namely, the ideal  $I^\vee$  is componentwise linear if and only if  $H_i(\mathbb{L}_{\bullet}^{<r>}(I^\vee)) = 0 \quad \forall i > 0$  and  $\forall r$ . The previous example has a trivial Lyubeznik table but  $R/I$  is not sequentially Cohen-Macaulay.

The simplest examples of ideals with non-trivial Lyubeznik table are minimal non-Cohen-Macaulay squarefree monomial ideals (see [55])

**Example:** The unique minimal non-Cohen-Macaulay squarefree monomial ideal of pure height two in  $R = k[x_1, \dots, x_n]$  is:

$$\mathfrak{a}_n = (x_1, x_3) \cap \dots \cap (x_1, x_{n-1}) \cap (x_2, x_4) \cap \dots \cap (x_2, x_n) \cap (x_3, x_5) \cap \dots \cap (x_{n-2}, x_n).$$

$$\bullet \mathfrak{a}_4 = (x_1, x_3) \cap (x_2, x_4).$$

We have  $H_{\mathfrak{a}_4}^2(R) \cong H_{(x_1, x_3)}^2(R) \oplus H_{(x_2, x_4)}^2(R)$  and  $H_{\mathfrak{a}_4}^3(R) \cong E_1$ . Thus its Lyubeznik table is

$$\Lambda(R/\mathfrak{a}_4) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

$$\bullet \mathfrak{a}_5 = (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_2, x_5) \cap (x_3, x_5).$$

We have  $H_{\mathfrak{a}_5}^3(R) \cong E_1$  and the hypercube associated to  $H_{\mathfrak{a}_5}^2(R)$  satisfy  $[H_{\mathfrak{a}_5}^2(R)]_\alpha \cong k$  for

$$\begin{aligned} \cdot \alpha &= (1, 0, 1, 0, 0), (1, 0, 0, 1, 0), (0, 1, 0, 1, 0), (0, 1, 0, 0, 1), (0, 0, 1, 0, 1) \\ \cdot \alpha &= (1, 1, 0, 1, 0), (1, 0, 1, 1, 0), (1, 0, 1, 0, 1), (0, 1, 1, 0, 1), (0, 1, 0, 1, 1) \end{aligned}$$

The complex associated to the hypercube is

$$0 \longleftarrow 0 \longleftarrow 0 \longleftarrow k^5 \xleftarrow{u_2} k^5 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$$

where the matrix corresponding to  $u_2$  is the rank 4 matrix:

$$\begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Thus its Lyubeznik table is

$$\Lambda(R/\mathfrak{a}_5) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$$

In general one gets

$$\Lambda(R/\mathfrak{a}_n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & & 0 & 0 & 1 \\ & & & \ddots & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & & 1 \end{pmatrix}$$

and the result agrees with [77, Cor. 5.5]

It is well-know that local cohomology modules as well as free resolutions depend on the characteristic of the base field, the most recurrent example being the Stanley-Reisner ideal associated to a minimal triangulation of  $\mathbb{P}_{\mathbb{R}}^2$ . Thus, Lyubeznik numbers also depend on the characteristic.

**Example:** Consider the ideal in  $R = k[x_1, \dots, x_6]$ :

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6)$$

The Lyubeznik table in characteristic zero and two are respectively:

$$\Lambda_{\mathbb{Q}}(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix} \quad \Lambda_{\mathbb{Z}/2\mathbb{Z}}(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$$

## 7. INJECTIVE RESOLUTION OF $H_I^r(R)$

The methods developed in the previous section allow us to describe the Bass numbers in the minimal  $\mathbb{Z}^n$ -graded injective resolution of a module with variation zero  $M$ . That

is:

$$\mathbb{I}^\bullet(M) : \quad 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} I^m \xrightarrow{d^m} \cdots,$$

where the  $j$ -th term is

$$I^j = \bigoplus_{\alpha \in \{0,1\}^n} E_\alpha^{\mu_j(\mathfrak{p}_\alpha, M)}.$$

In particular we are able to compute the  $\mathbb{Z}^n$ -graded injective dimension of  $M$  or the  $\mathbb{Z}^n$ -graded small support. The aim of this section is to take a close look to the structure of this minimal injective resolution.

**7.1. Injective dimension.** Let  $(R, \mathfrak{m}, k)$  be a local ring and let  $M$  be an  $R$ -module. Bass numbers of finitely generated modules are known to satisfy the following properties:

- 1)  $\mu_i(\mathfrak{p}, M) < +\infty, \forall i, \forall \mathfrak{p} \in \text{Supp}_R M$
- 2) Let  $\mathfrak{p} \subseteq \mathfrak{q} \in \text{Spec} R$  such that  $\text{ht}(\mathfrak{q}/\mathfrak{p}) = s$ . Then

$$\mu_i(\mathfrak{p}, M) \neq 0 \implies \mu_{i+s}(\mathfrak{q}, M) \neq 0.$$

- 3)  $\text{id}_R M := \sup\{i \in \mathbb{Z} \mid \mu_i(\mathfrak{m}, M) \neq 0\}$
- 4)  $\text{depth}_R M \leq \dim_R M \leq \text{id}_R M$

When  $M$  is not finitely generated, similar properties for Bass numbers are known for some special cases. A. M. Simon [78] proved that properties 2) and 3) are still true for complete modules and M. Hellus [41] proved that  $\dim_R M \leq \text{id}_R M$  for cofinite modules.

For the case of local cohomology modules, C. Huneke and R. Sharp [45] and G. Lyubeznik [56, 57], proved that for a regular local ring  $(R, \mathfrak{m}, k)$  containing a field  $k$ :

- 1)  $\mu_i(\mathfrak{p}, H_I^r(R)) < +\infty, \forall i, \forall r, \forall \mathfrak{p} \in \text{Supp}_R H_I^r(R)$
- 4')  $\text{id}_R H_I^r(R) \leq \dim_R \text{Supp}_R H_I^r(R)$

In this section we want to study property 2) for the particular case of local cohomology modules supported on monomial ideals and give a sharper bound to 4') in terms of the small support.

**Proposition 7.1.** *Let  $I \subseteq R = k[x_1, \dots, x_n]$  be a squarefree monomial ideal and set*

$$s := \max\{i \in \mathbb{Z}_{\geq 0} \mid \mu_i(\mathfrak{p}_\alpha, H_I^r(R)) \neq 0\}$$

*for all prime ideals  $\mathfrak{p}_\alpha \in \text{Supp}_R H_I^r(R)$  such that  $|\alpha| = n - 1$ . Then  $\mu_t(\mathfrak{m}, H_I^r(R)) = 0$   $\forall t > s + 1$ .*

Therefore we get the main result of this section:

**Theorem 7.2.** *Let  $I \subseteq R = k[x_1, \dots, x_n]$  be a squarefree monomial ideal. Then,  $\forall r$  we have*

$$*\text{id}_R H_I^r(R) \leq \dim_R * \text{supp}_R H_I^r(R)$$

**Remark 7.3.** Using [35, Thm. 1.2.3] we also have

$$\mathrm{id}_R H_I^r(R) \leq \dim_R \mathrm{supp}_R H_I^r(R)$$

but one must be careful with the ring  $R$  we consider. In the example above we have:

- $\mathrm{id}_R H_I^3(R) = \mathrm{id}_R H_I^3(R) < \dim_R \mathrm{supp}_R H_I^3(R)$  if  $R = k[[x_1, \dots, x_n]]$
- $\mathrm{id}_R H_I^3(R) < \mathrm{id}_R H_I^3(R) = \dim_R \mathrm{supp}_R H_I^3(R)$  if  $R = k[x_1, \dots, x_n]$ .

**Remark 7.4.** Consider the largest chain of prime face ideals  $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \dots \subseteq \mathfrak{p}_n$  in the small support of a local cohomology module  $H_I^r(R)$ . In these best case scenario we have a version of property 2) that we introduced at the beginning of this section that reads off as:

- $\mu_0(\mathfrak{p}_0, H_I^r(R)) = 1$  and  $\mu_j(\mathfrak{p}_0, H_I^r(R)) = 0 \ \forall j > 0$ .
- $\mu_i(\mathfrak{p}_i, H_I^r(R)) \neq 0$  and  $\mu_j(\mathfrak{p}_i, H_I^r(R)) = 0 \ \forall j > i$ , for all  $i = 1, \dots, n$ .

Then:

- i)  $\mathrm{id}_R H_I^r(R) = \dim_R(\mathrm{supp}_R H_I^r(R))$  if and only if this version of property 2) is satisfied.
- ii)  $\mathrm{id}_R H_I^r(R) = \dim_R \mathrm{Supp}_R H_I^r(R)$  if and only if this version of property 2) is satisfied and  $\mathfrak{m} \in \mathrm{supp}_R H_I^r(R)$ .

This sheds some light on the examples treated in [41] where the question whether the equality  $\mathrm{id}_R H_I^r(R) = \dim_R H_I^r(R)$  holds is considered. On the other end of possible cases we may have:

- $\mu_0(\mathfrak{p}_0, H_I^r(R)) = \mu_0(\mathfrak{p}_n, H_I^r(R)) = 1$  and  $\mu_j(\mathfrak{p}_0, H_I^r(R)) = \mu_j(\mathfrak{p}_n, H_I^r(R)) = 0 \ \forall j > 0$ .

Notice that in this case the same property holds for any prime ideal  $\mathfrak{p}_i$  in the chain. In particular all the primes in the chain are associated primes of  $H_I^r(R)$ .

## Appendix: $\mathbb{Z}^n$ -graded free and injective resolutions

The theory of  $\mathbb{Z}^n$ -graded rings and modules is analogous to that of  $\mathbb{Z}$ -graded rings and modules. The aim of this appendix is to fix the notation that we use throughout this work. For a detailed exposition of these topics we refer to [18], [35], [65].

In this work, we will only treat the special case of modules over the polynomial ring  $R = k[x_1, \dots, x_n]$ , where  $k$  is a field of any characteristic and  $x_1, \dots, x_n$  are independent variables. Let  $\varepsilon_1, \dots, \varepsilon_n$  be the canonical basis of  $\mathbb{Z}^n$ . Then, the ring  $R$  has a natural  $\mathbb{Z}^n$ -graduation given by  $\deg(x_i) = \varepsilon_i$ . Henceforth, the term graded will always mean  $\mathbb{Z}^n$ -graded. If  $M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_\alpha$  is a graded  $R$ -module and  $\beta \in \mathbb{Z}^n$ , as usual we denote by  $M(\beta)$  the shifted graded  $R$ -module whose underlying  $R$ -module structure is the same as that of



$M$  and where the grading is given by  $(M(\beta))_\alpha = M_{\beta+\alpha}$ . In particular, the free  $R$ -module of rank one generated in degree  $\alpha \in \mathbb{Z}^n$  is  $R(-\alpha)$ .

**7.2. Free resolutions.** The minimal graded free resolution of a monomial ideal  $J \subseteq R$  is an exact sequence of free  $\mathbb{Z}^n$ -graded  $R$ -modules:

$$\mathbb{L}_\bullet(J) : 0 \longrightarrow L_m \xrightarrow{d_m} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow J \longrightarrow 0$$

where the  $j$ -th term is of the form

$$L_j = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{j,\alpha}(J)},$$

and the matrices of the morphisms  $d_j : L_j \longrightarrow L_{j-1}$  do not contain invertible elements. From this expression we can get the following:

- The **projective dimension** of  $J$ , denoted  $\text{pd}(J)$ , is the greatest homological degree in the resolution. Namely

$$\text{pd}(J) := \max\{j \mid L_j \neq 0\}.$$

- The  $\mathbb{Z}^n$ -graded **Betti numbers** of  $J$  are the invariants defined by  $\beta_{j,\alpha}(J)$ . Betti numbers can also be described as:

$$\beta_{j,\alpha}(J) = \dim_k \text{Tor}_j^R(J, k)_\alpha.$$

- The **Castelnuovo-Mumford regularity** of  $J$  denoted  $\text{reg}(J)$  is

$$\text{reg}(J) := \max\{|\alpha| - j \mid \beta_{j,\alpha}(J) \neq 0\}.$$

- Given an integer  $r$ , the  **$r$ -linear strand** of  $\mathbb{L}_\bullet(J)$  is the complex:

$$\mathbb{L}_\bullet^{<r>}(J) : 0 \longrightarrow L_m^{<r>} \xrightarrow{d_m^{<r>}} \cdots \longrightarrow L_1^{<r>} \xrightarrow{d_1^{<r>}} L_0^{<r>} \longrightarrow 0,$$

where

$$L_j^{<r>} = \bigoplus_{|\alpha|=j+r} R(-\alpha)^{\beta_{j,\alpha}(J)},$$

and the differentials  $d_j^{<r>} : L_j^{<r>} \longrightarrow L_{j-1}^{<r>}$  are the corresponding components of  $d_j$ .

The study of Betti numbers of any monomial ideal can be reduced to the understanding of the squarefree monomial case but, even in this case, describing the minimal free resolution may be hard. This has been a very active area of research so we can not cover all the existent literature in this survey. We only want to point out that, apart from describing the minimal free resolution of particular cases of monomial ideals, some succesful lines of research are:

- Construct possibly non-minimal free resolutions. The most famous one being:

- *Taylor resolution* [82]: Let  $\{\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_m}\}$  be a set of generators of  $I$ . Let  $L$  be the free  $R$ -module of rank  $m$  generated by  $e_1, \dots, e_m$ . The Taylor complex  $\mathbb{T}_\bullet(R/I)$  is of the form:

$$\mathbb{T}_\bullet(R/I) : 0 \longrightarrow L_m \xrightarrow{d_m} \dots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow 0 ,$$

where  $L_j = \bigwedge^j L$  is the  $j$ -th exterior power of  $L$  and if  $\text{lcm}$  denotes the least common multiple, then the differentials  $d_j$  are defined as

$$d_j(e_{i_1} \wedge \dots \wedge e_{i_j}) = \sum_{1 \leq k \leq j} (-1)^k \frac{\text{lcm}(\mathbf{x}^{\alpha_{i_1}}, \dots, \mathbf{x}^{\alpha_{i_j}})}{\text{lcm}(\mathbf{x}^{\alpha_{i_1}}, \dots, \widehat{\mathbf{x}^{\alpha_{i_k}}}, \dots, \mathbf{x}^{\alpha_{i_j}})} e_{i_1} \wedge \dots \wedge \widehat{e_{i_k}} \wedge \dots \wedge e_{i_j}.$$

Some subcomplexes of the Taylor resolution as the *Lyubeznik resolution* [54] or the *Scarf complex* [10] also have a prominent role in the theory. These are examples of *simplicial resolutions*. The idea behind the concept of simplicial (resp. cellular, CW) resolution introduced in [10] (see also [11], [47]) is to associate to a free resolution of an ideal a simplicial complex (resp. regular cell complex, CW-complex) that carries in its structure the algebraic structure of the free resolution.

- Relating the  $\mathbb{Z}^n$ -graded Betti numbers of monomial ideals to combinatorial objects described by the generators of the monomial ideal. Here we highlight:

- *Hochster's formula*: Via the Stanley-Reisner correspondence we associate a simplicial complex  $\Delta$  to our squarefree monomial ideal  $I = I_\Delta$ . Then, for a given face  $\sigma_\alpha \in \Delta$  consider the restriction  $\Delta_\alpha := \{\tau \in \Delta \mid \tau \in \sigma_\alpha\}$ . Then we have

$$\beta_{i,\alpha}(I_\Delta) = \dim_k \tilde{H}_{|\alpha|-i-2}(\Delta_\alpha; k)$$

where  $\tilde{H}(-; k)$  denotes reduced simplicial homology. Here we agree that the reduced homology with coefficients in  $k$  of the empty simplicial complex is  $k$  in degree  $-1$  and zero otherwise.

- *LCM-lattice* [33]: The lcm-lattice of a monomial ideal is the poset  $P(I)$  of the least common multiples of the generators of the ideal. To any poset one can associate the *order complex*, this is a simplicial complex which has as vertices the elements of the poset and where a set of vertices  $p_0, \dots, p_r$  determines a  $r$ -dimensional simplex if  $p_0 < \dots < p_r$ . Now we define  $K(> p)$  to be the simplicial complex attached to the subposet  $\{q \in P(I) \mid q > p\}$ . Then,

$$\beta_{i,p}(I_\Delta) = \dim_k \tilde{H}_{i-2}(K(> p); k)$$

It follows that if two ideals have isomorphic lcm-lattices then they have the same Betti numbers up to some relabeling of the degrees.

**7.3. Injective resolutions.** If  $M$  is a  $\mathbb{Z}^n$ -graded module one can define its  $\mathbb{Z}^n$ -graded injective envelope  ${}^*E(M)$ . Therefore, the category  ${}^*\text{Mod}(R)$  of  $\mathbb{Z}^n$ -graded modules has

enough injectives and a  $\mathbb{Z}^n$ -graded version of the Matlis–Gabriel theorem holds: The indecomposable injective objects of  ${}^*\text{Mod}(R)$  are the shifted injective envelopes  ${}^*E(R/\mathfrak{p}_\alpha)(\beta)$ , where  $\mathfrak{p}_\alpha$  is a face ideal of  $R$  and  $\beta \in \mathbb{Z}^n$ , and every graded injective module is isomorphic to a unique (up to order) direct sum of indecomposable injectives.

The minimal graded injective resolution of  $\mathbb{Z}^n$ -graded  $R$ -module  $M$  is a sequence:

$$\mathbb{I}^\bullet(M) : 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \longrightarrow I^m \xrightarrow{d^m} \cdots,$$

exact everywhere except the 0-th step such that  $M = \text{Ker}(d^0)$ , the  $j$ -th term is

$$I^j = \bigoplus_{\alpha \in \mathbb{Z}^n} {}^*E(R/\mathfrak{p}_\alpha)(\beta)^{\mu_j(\mathfrak{p}_\alpha, M)},$$

and  $I^j$  is the injective envelope of  $\text{Ker } d^j$ . From this expression we can get the following:

- The  **$\mathbb{Z}^n$ -graded injective dimension** of  $M$ , denoted  ${}^*\text{id}_R(M)$ , is the greatest cohomological degree in the minimal graded injective resolution. Namely

$${}^*\text{id}_R(M) = \max\{j \mid I^j \neq 0\}.$$

**Remark 7.5.** *The notation  ${}^*\text{id}_R$  usually refers to the  $\mathbb{Z}$ -graded injective dimension but we use the same notation for the  $\mathbb{Z}^n$ -graded injective dimension as in [87]. The reader must be aware that both concepts are different but in this work no confusion is possible since we only consider the  $\mathbb{Z}^n$ -graded case.*

- The **Bass numbers** of  $M$  are the invariants defined by  $\mu_j(\mathfrak{p}_\alpha, M)$ . By using the results of [35], these numbers are equal to the usual Bass numbers that appear in the minimal injective resolution of  $M$ . So, they can also be computed as

$$\mu_j(\mathfrak{p}_\alpha, M) = \dim_{k(\mathfrak{p}_\alpha)} \text{Ext}_R^j(k(\mathfrak{p}_\alpha), M_{\mathfrak{p}_\alpha}).$$

If we want to compute the Bass numbers with respect to any prime ideal we have to refer to the result of S. Goto and K. I. Watanabe [35, Thm. 1.2.3]. Namely, given any prime ideal  $\mathfrak{p} \in \text{Spec } R$ , let  $\mathfrak{p}_\alpha$  be the largest face ideal contained in  $\mathfrak{p}$ . If  $\text{ht}(\mathfrak{p}/\mathfrak{p}_\alpha) = s$  then  $\mu_p(\mathfrak{p}_\alpha, M) = \mu_{p+s}(\mathfrak{p}, M)$ . Notice that in general we have  ${}^*\text{id}_R M \leq \text{id}_R M$ .

- The **small support** of  $M$  introduced by H. B. Foxby [27] is defined as

$$\text{supp}_R M := \{\mathfrak{p} \in \text{Spec } R \mid \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty\},$$

where  $\text{depth}_R M := \inf\{i \in \mathbb{Z} \mid \text{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\} = \inf\{i \in \mathbb{Z} \mid \mu_i(\mathfrak{m}, M) \neq 0\}$ . In terms of Bass numbers we have that  $\mathfrak{p} \in \text{supp}_R M$  if and only if there exists some integer  $i \geq 0$  such that  $\mu_i(\mathfrak{p}, M) \neq 0$ . It is also worth to point out that  $\text{supp}_R M \subseteq \text{Supp}_R M$ , and equality holds when  $M$  is finitely generated. It follows that in general we have  $\dim_R \text{supp}_R M \leq \dim_R \text{Supp}_R M$ .

We can also define the  $\mathbb{Z}^n$ -graded small support that we denote  ${}^*\text{supp}_R M$  as the set of face ideals in the support of  $M$  that at least have a Bass number different from zero.

- Given an integer  $r$ , the  $r$ -**linear strand** of  $\mathbb{I}_\bullet(M)$  is the complex:

$$\mathbb{I}_\bullet^{<r>}(M) : 0 \longrightarrow I_0^{<r>} \longrightarrow I_1^{<r>} \longrightarrow \cdots \longrightarrow I_m^{<r>} \longrightarrow 0 ,$$

where

$$I_j^{<r>} = \bigoplus_{|\alpha|=j+r} {}^*E(R/\mathfrak{p}_\alpha)(\beta)^{\mu_j(\mathfrak{p}_\alpha, M)},$$

**7.4. Monomial matrices.** The concept of *monomial matrices* was coined by E. Miller in [64] (see also [65]) to deal with maps of  $\mathbb{Z}^n$ -graded free, injective or flat modules in an unified way.

A matrix whose  $(p, q)$ -entry is of the form  $\lambda_{pq}\mathbf{x}^{\beta_{pq}}$ , where  $\beta_{pq} \in \mathbb{Z}^n$ , defines a map of  $\mathbb{Z}^n$ -graded free modules but we also have to keep track of the degrees of the generators in the source and the target of the matrix to determine the map uniquely. Once we do this, we can simplify the notation just using the scalars  $\lambda_{pq}$  as entries of our matrix

$$\begin{matrix} & \beta_{\cdot 1} & \cdots & \beta_{\cdot q} & \cdots \\ \beta_{1 \cdot} & & & & \\ \vdots & & & & \\ \beta_{p \cdot} & & & \lambda_{pq} & \\ \vdots & & & & \end{matrix} \left( \begin{matrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{matrix} \right)$$

The nice idea in [64] is that, modifying conveniently the notation for the source and the target, one can use the same kind of matrices to describe maps between injective modules or flat modules. Now we allow  $\beta_{pq} \in (\mathbb{Z} \cup *)^n$  where  $*$  behaves like  $-\infty$  except for  $-1 \cdot * = *$ .

There are several technical questions to be addressed for these monomial matrices to work out but we will skip the details. In this work we only have to have in mind that the matrices that describe a free resolution or a injective resolution have scalar entries with the appropriate source and target labels. This will be also true for the maps describing the Čech complex.

## Tutorial (joint with O. Fernández-Ramos)

Over the last twenty years there were many advances made in the computational theory of  $D$ -modules. Nowadays, the most common computer algebra systems<sup>8</sup> such as **Macaulay2** or **Singular** have important available packages for working with  $D$ -modules. In particular, the package **D-modules** [52] for **Macaulay 2** [37] developed by A. Leykin and H. Tsai contains an implementation of the algorithms given by U. Walther [84] and T. Oaku and N. Takayama [69] to compute local cohomology modules. For a quick

---

<sup>8</sup>CoCoA is still working on that.

overview of this general approach we recommend the interested reader to take a look at [85], [51]. We have to point out that the algorithm to compute Lyubeznik numbers given by U. Walther [84] is not available in this package (see [7] for an alternative approach).

The complexity of the algorithm turns out to be a major drawback when trying to compute even some basic examples such as local cohomology modules supported on monomial ideals. It should be noticed that F. Barkats [9] gave an algorithm to compute a presentation of this kind of modules. However she was only able to compute effectively examples in the polynomial ring  $k[x_1, \dots, x_6]$ .

In order to prepare the Tutorials of the course given at "MONICA: MONomial Ideals, Computations and Applications" we implemented our own functions to compute the characteristic cycle of local cohomology modules supported on monomial ideals and Lyubeznik numbers as well. We finally decided to use `Macaulay2` since it already provides some packages to deal with edge ideals and simplicial complexes. In fact, our functions call other functions from the `SimplicialComplexes` package written by S. Popescu, G. G. Smith and M. Stillman. Also useful in this context, in order to experiment with many examples, is the `EdgeIdeals` package written by C. Francisco, A. Hoefel and A. Van Tuyl. For more information on this last package we recommend to take a look at the notes provided by A. Van Tuyl in Section 4 of this volume.

The following functions are not included in any version of `Macaulay2`. We collected them in the file `LCfunctions.v4.m2` that we posted in

[http://monica.unirioja.es/conference/monica\\_program.html](http://monica.unirioja.es/conference/monica_program.html)

- **multCCLC**: It computes the multiplicities and the components of the characteristic variety of a local cohomology module supported on a squarefree monomial ideal. The input is just the squarefree monomial ideal and it returns as output a hashtable whose keys correspond to the cohomology degrees and the values are lists of pairs. Each pair consists of the multiplicity and the corresponding component of the characteristic variety. Each component is expressed as the variety of the defining ideal. For a more visual output we collect the varieties by their corresponding height using the command `fancyOut` with the output of `multCCLC` as input.
- **lyubeznikTable**: It computes the Lyubeznik table of a squarefree monomial ideal. First we have to compute the minimal free resolution of its Alexander dual ideal. Then, the command `linearStrands` extracts the linear strands of this resolution and finally, the command `lyubeznikTable` computes the homology groups of the linear strands and displays the corresponding Lyubeznik numbers in a BettiTally as output.

## 8. GETTING STARTED

Before starting you will have to save the source code in your working directory. It can be obtained using the command `path` in a running session of `Macaulay2`:

```
Macaulay2, version 1.4
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone

i1 : path

o1 = {./, .Macaulay2/code/, .Macaulay2/local/share/Macaulay2/,
-----
     .Macaulay2/local/common/share/Macaulay2/, /usr/share/Macaulay2/}

o1 : List
```

where `./` means the directory from where you run `Macaulay2`. You can also check it out using `currentDirectory()`. Once we have this package installed we are ready to start our session. First we will have to load it

```
i2 : load "LCfunctionsv4.m2"
```

Then we introduce our favorite polynomial ring, but we have to make sure that we give the appropriate  $\mathbb{Z}^n$ -grading to the variables.

```
i3 : R=QQ[x_1..x_5,DegreeRank=>5]

o3 = R

o3 : PolynomialRing

i4 : R_0

o4 = x
     1

o4 : R

i5 : degree R_0

o5 = {1, 0, 0, 0, 0}

o5 : List
```

Now we introduce a squarefree monomial ideal. Just for completeness we check out its minimal primary decomposition. We also compute its Alexander dual ideal since we will need it later on.

```

i6 : I=monomialIdeal (x_1*x_2,x_2*x_4,x_1*x_5,x_3*x_4*x_5)

o6 = monomialIdeal (x x , x x , x x , x x x )
                  1 2   2 4   1 5   3 4 5

o6 : MonomialIdeal of R

i7 : primaryDecomposition I

o7 = {monomialIdeal (x , x ), monomialIdeal (x , x ), monomialIdeal (x , x , x )}
      1 4           2 5           1 2 3

o7 : List

i8 : Idual=dual I

o8 = monomialIdeal (x x x , x x , x x )
                  1 2 3   1 4   2 5

o8 : MonomialIdeal of R

```

The characteristic cycle of the corresponding local cohomology modules are

```

i9 : multCCLC I

o9 = HashTable{2 => {{1, (x , x )}, {1, (x , x )}}
                  1 4           2 5
              3 => {{1, (x , x , x )}, {1, (x , x , x , x )}, {1, (x , x , x , x )}, {1, (x , x , x , x )},
                  1 2 3           1 2 3 5           1 2 3 4           1 2 4 5
                  {1, (x , x , x , x , x )}}
                  1 2 3 4 5
              4 => {}

o9 : HashTable

i10 : fancyOut oo

o10 = HashTable{H^2 => {{(1, V(x_1,x_4))}}
                  {(1, V(x_2,x_5))}
              H^3 => {{(1, V(x_1,x_2,x_3))}, {(1, V(x_1,x_2,x_3,x_5))}, {(1, V(x_1,x_2,x_3,x_4,x_5))}}
                  {(1, V(x_1,x_2,x_3,x_4))}
                  {(1, V(x_1,x_2,x_4,x_5))}

o10 : HashTable

```

Notice that we obtain two local cohomology modules different from zero and we can easily describe the support of these modules and their dimension. We can also compare the multiplicities with the Betti numbers of the Alexander dual ideal.

```

i11 :

```

```

r= res Idual

      1      3      3      1
o11 = R <-- R <-- R <-- R <-- 0

      0      1      2      3      4

o11 : ChainComplex

i12 : LS=linearStrands r

      2
o12 = HashTable{2 => R      }

      1
      1      3      1
      3 => R <-- R <-- R

      1      2      3
      4 => 0

o12 : HashTable

i13 : B=betti r

      0 1 2 3
o13 = total: 1 3 3 1
      0: 1 . . .
      1: . 2 . .
      2: . 1 3 1

o13 : BettiTally

```

We already computed the linear strands of the minimal free resolution of the Alexander dual ideals so we are ready to compute the Lyubeznik table

```

i14 : T=lyubeznikTable LS

      0 1 2 3
o14 = total: . . 1 2
      0: . . . .
      1: . . 1 .
      2: . . . .
      3: . . . 2

o14 : BettiTally

```



## 9. TUTORIAL

Once we get acquainted with the use of these functions we propose several exercises. This is just a small sample of questions. We encourage the readers to experiment with different families of examples and come up with their own formulas for Lyubeznik numbers or, in general, Bass numbers. The question on how to find a general description of the injective resolution, i.e. Bass numbers and maps between injective modules, for any ideal might be too difficult. As in the case of free resolutions it would be interesting to study the different linear strands in the injective resolution of local cohomology modules.

A recurrent topic in recent years has been to attach a cellular structure to the free resolution of a monomial ideal. In general this can not be done as it is proved in [83] but there are large families of ideals having a cellular resolution. Using the dictionary we described in Section 5 we can translate the same questions to Lyubeznik numbers. In particular it would be interesting to find cellular structures on the linear strands of a free resolution so one can give a topological description of Lyubeznik numbers.

**Exercise 9.0.1.** *Given the ideals*

- $I_1 = (x_1x_4, x_2x_3),$
- $I_2 = (x_1x_2, x_1x_3, x_2x_4)$
- $I_3 = (x_1x_2x_3, x_1x_4, x_2x_4)$

- a) *Compute the characteristic cycle of the corresponding local cohomology modules.*
- b) *Describe the support of these modules and compute its dimension.*
- c) *Are the corresponding Stanley-Reisner rings Cohen-Macaulay, Gorenstein or Buchsbaum?*

**Exercise 9.0.2.** *Describe the  $n$ -hypercube of the local cohomology modules supported on the ideals of the previous exercise.*

**Exercise 9.0.3.** *Given the ideal*

$$I = (x_1, x_2) \cap (x_3, x_4) \cap (x_5, x_6) \cap (x_7, x_8) \cap (x_9, x_1) \cap (x_9, x_2) \cap (x_9, x_3) \cap (x_9, x_4) \cap \\ \cap (x_9, x_5) \cap (x_9, x_6) \cap (x_9, x_7) \cap (x_9, x_8)$$

- a) *Compute the characteristic cycle of the corresponding local cohomology modules and construct the corresponding table  $\Gamma(R/I)$ .*
- b) *Compute the Betti table of the Alexander dual ideal  $I^\vee$ .*
- c) *Is  $I$  sequentially Cohen-Macaulay?*
- d) *Compute its Lyubeznik table.*

*Hint: The EdgeIdeals package has the isSCM command to check out the sequentially Cohen-Macaulay property.*

**Exercise 9.0.4.** Let  $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1x_5)$  be the edge ideal of a 4-cycle with a whisker

- a) Is  $I$  sequentially Cohen-Macaulay?
- b) Compute its Lyubeznik table.

**Exercise 9.0.5.** Consider your favorite sequentially Cohen-Macaulay ideal and compute its Lyubeznik table.

**Exercise 9.0.6.** Consider the ideal  $I = (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_2, x_5) \cap (x_3, x_5)$ .

- a) Compute all the Bass numbers of the local cohomology module  $H_I^2(R)$ .
- b) Is the injective resolution linear?

**Exercise 9.0.7.** Find examples of local cohomology modules  $M = H_I^r(R)$  such that their Bass numbers satisfy

- $\mu_s(\mathfrak{p}_\alpha, M) \neq 0$  and  $\mu_s(\mathfrak{m}, M) = 0$ ,  $\mu_{s+1}(\mathfrak{m}, M) \neq 0$
- $\mu_s(\mathfrak{p}_\alpha, M) \neq 0$  and  $\mu_s(\mathfrak{m}, M) \neq 0$ ,  $\mu_{s+1}(\mathfrak{m}, M) = 0$
- $\mu_s(\mathfrak{p}_\alpha, M) \neq 0$  and  $\mu_s(\mathfrak{m}, M) \neq 0$ ,  $\mu_{s+1}(\mathfrak{m}, M) \neq 0$
- $\mu_s(\mathfrak{p}_\alpha, M) \neq 0$  and  $\mu_s(\mathfrak{m}, M) = 0$ ,  $\mu_{s+1}(\mathfrak{m}, M) = 0$

where  $\mathfrak{p}_\alpha \subseteq \mathfrak{m}$  is a face ideal such that  $\text{ht}(\mathfrak{m}/\mathfrak{p}_\alpha) = 1$

**Exercise 9.0.8.** Compute all the Bass numbers of the following modules with variation zero in  $R = k[x_1, \dots, x_5]$

- $M = H_{(x_1, x_2, x_3, x_4)}^4(R) \oplus E_{(0,0,1,1,1)}$
- $M = H_I^3(R)$ , where  $I = (x_1x_4, x_2x_3, x_3x_4, x_1x_2x_5)$

Are these modules isomorphic?

**Exercise 9.0.9.** Find a monomial ideal  $I$  such that  $\text{id}_R H_I^r(R) < \dim_R \text{Supp}_R H_I^r(R)$  for some  $r$ .

**Exercise 9.0.10.** Find a formula for the Lyubeznik table of the Alexander dual of the edge ideals  $I(C_n)$  of the cycle graph  $C_n$ ,  $n \geq 3$

**Exercise 9.0.11.** The same as in the previous exercise but for the complement  $C_n^c$  of the cycle graph  $C_n$ ,  $n \geq 3$ .

**Exercise 9.0.12.** Compute the Lyubeznik table of the ideal associated to a minimal triangulation of  $\mathbb{P}_{\mathbb{R}}^2$  when the characteristic of the field is 0 and 2.

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6)$$

**Exercise 9.0.13.** Is there any ideal such that the local cohomology modules depend on the characteristic of the field but the Lyubeznik numbers do not?

*Hint:* Modify conveniently the ideal associated to a minimal triangulation of  $\mathbb{P}_{\mathbb{R}}^2$ .

## REFERENCES

- [1] J. Àlvarez Montaner, Characteristic cycles of local cohomology modules of monomial ideals, J. Pure Appl. Algebra **150** (2000), 1–25.
- [2] J. Àlvarez Montaner, *Characteristic cycles of local cohomology modules of monomial ideals II*, J. Pure Appl. Algebra **192** (2004), 1–20.
- [3] J. Àlvarez Montaner, *Some numerical invariants of local rings*, Proc. Amer. Math. Soc. **132** (2004), 981–986.
- [4] J. Àlvarez Montaner, *Operations with regular holonomic  $D$ -modules with support a normal crossing*, J. Symb. Comp. **40**, (2005), 999–1012.
- [5] J. Àlvarez Montaner, R. García López and S. Zarzuela, *Local cohomology, arrangements of subspaces and monomial ideals*, Adv. Math. **174** (2003), 35–56.
- [6] J. Àlvarez Montaner and S. Zarzuela, *Linearization of local cohomology modules*, Contem. Math. **331**, 1–11, Amer. Math. Soc. (2003).
- [7] J. Àlvarez Montaner and A. Leykin, *Computing the support of local cohomology modules*, J. Symb. Comp. **41**, (2006), 1328–1344.
- [8] J. Àlvarez Montaner and A. Vahidi, *Lyubeznik numbers of monomial ideals*, to appear in Trans. Amer. Math. Soc. arXiv/1107.5230.
- [9] F. Barkats, *Calcul effectif de groupes de cohomologie locale à support dans des idéaux monomiaux*, Ph.D. Thesis, Univ. Nice-Sophia Antipolis, 1995.
- [10] D. Bayer, I. Peeva and B. Sturmfels, *Monomial resolutions*, Math. Res. Lett., **5** (1998), 31–46.
- [11] D. Bayer and B. Sturmfels, *Cellular resolutions of monomial modules*, J. Reine Angew. Math., **502** (1998), 123–140.
- [12] J. E. Björk, *Rings of differential operators*, North Holland Mathematics Library, Amsterdam, 1979.
- [13] M. Blickle, *Lyubeznik’s numbers for cohomologically isolated singularities*, J. Algebra **308** (2007), 118–123.
- [14] M. Blickle, R. Bondu, *Local cohomology multiplicities in terms of étale cohomology*, Ann. Inst. Fourier **55** (2005), 2239–2256.
- [15] R. Boldini, *Critical cones of characteristic varieties*, to appear in Trans. Amer. Math. Soc., arXiv:1008.0292
- [16] A. Borel, *Algebraic  $D$ -modules*, Perspectives in Mathematics, Academic Press, 1987.
- [17] M. P. Brodmann and R. Y. Sharp, *Local cohomology, An algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics **60**, Cambridge University Press, 1998.
- [18] W. Bruns and J. Herzog, *Cohen Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, 1993.
- [19] S. C. Coutinho, *A primer of algebraic  $\mathcal{D}$ -modules*, London Mathematical Society Student Texts, Cambridge University Press, 1995.
- [20] K. Dalili and M. Kummini, *Dependence of Betti numbers on characteristic* arXiv:1009.4243.
- [21] J. A. Eagon and V. Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra, **130** (1998), no. 3, 265–275.
- [22] D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, GTM **150**, Springer-Verlag, 1995.
- [23] D. Eisenbud, G. Fløystad and F.O. Schreyer, *Sheaf cohomology and free resolutions over exterior algebras*, Trans. Amer. Math. Soc., **355** (2003), 4397–4426.
- [24] D. Eisenbud, M. Mustață, and M. Stillman, *Cohomology on toric varieties and local cohomology with monomial supports*, J. Symbolic Comput., **29** (2000), 583–600.
- [25] S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129** (1990), 1–25.
- [26] G. Fatabbi, *On the resolution of ideals of fat points*, J. Algebra **242** (2001), 92–108.

- [27] H.B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), 149–172.
- [28] C. Francisco, H. T. Hà and A. Van Tuyl, *Splittings of monomial ideals*, Proc. Amer. Math. Soc. **137** (2009), 3271–3282.
- [29] O. Gabber, *The integrability of the Characteristic Variety*, Amer. J. of Math. **103** (1981), 445–468.
- [30] A. Galligo, M. Granger and Ph. Maisonobe, *D-modules et faisceaux pervers dont le support singulier est un croisement normal*, Ann. Inst. Fourier **35** (1985), 1–48.
- [31] A. Galligo, M. Granger and Ph. Maisonobe, *D-modules et faisceaux pervers dont le support singulier est un croisement normal. II*, in “Differential systems and singularities (Luminy, 1983)” Astérisque **130** (1985), 240–259.
- [32] R. Garcia and C. Sabbah, *Topological computation of local cohomology multiplicities*, Collect. Math. **49** (1998), 317–324.
- [33] V. Gasharov, I. Peeva and V. Welker, *The lcm-lattice in monomial resolutions*, Math. Res. Lett. **6** (1999), 521–532.
- [34] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Ergebnisse Series **14**, Springer-Verlag, 1988.
- [35] S. Goto and K. Watanabe, *On Graded Rings, II ( $\mathbb{Z}^n$ - graded rings)*, Tokyo J. Math. **1**(2) (1978), 237–261.
- [36] H. G. Gräbe, *The canonical module of a Stanley-Reisner ring*, J. Algebra **86** (1984), 272–281.
- [37] D. Grayson and M. Stillman, *Macaulay 2*. Available at: <http://www.math.uiuc.edu/Macaulay2>.
- [38] A. Grothendieck, *Local Cohomology*, Lect. Notes in Math. **41**, Springer-Verlag, 1967.
- [39] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas*, Publ. Math. I.H.E.S. **32**, 1967.
- [40] H. T. Hà and A. Van Tuyl, *Splittable ideals and the resolution of monomial ideals*, J. Algebra **309** (2007), 405–425.
- [41] M. Hellus, *A note on the injective dimension of local cohomology modules*, Proc. Amer. Math. Soc. **136** (2008), 2313–2321.
- [42] J. Herzog and T. Hibi, *Componentwise linear ideals*, Nagoya Math. J. **153** (1999), 141–153.
- [43] J. Herzog and S. Iyengar, *Koszul modules*, J. Pure Appl. Algebra **201** (2005), 154–188.
- [44] C. Huneke, *Problems on local cohomology modules*, in: Free resolutions in commutative algebra and algebraic geometry (Sundance, 1990) Research Notes in Math. **2**, Boston, MA, Jones and Barthlett Publishers (1994), 93–108.
- [45] C. Huneke and R. Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), 765–779.
- [46] S. Iyengar, G. Leuschke, A. Leykin, C. Miller, E. Miller, A. Singh and U. Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, 87. American Mathematical Society, 2007.
- [47] M. Jöllenbeck and V. Welker, *Minimal resolutions via algebraic discrete Morse theory*. Mem. Amer. Math. Soc. **197** (2009).
- [48] F. Kirwan, *An introduction to intersection homology theory*, Pitman Research Notes in Mathematics, Longman Scientific and Technical, 1988.
- [49] S. Khoroshkin, *D-modules over arrangements of hyperplanes*, Comm. in Alg. **23** (1995), 3481–3504.
- [50] S. Khoroshkin, A. Varchenko, *Quiver D-modules and homology of local systems over arrangements of hyperplanes*, Inter. Math. Res. Papers, 2006.
- [51] A. Leykin, *D-modules for Macaulay 2*. In Mathematical Software: ICMS 2002, pages 169–179. World Scientific, 2002.
- [52] A. Leykin, M. Stillman and H. Tsai, *D-modules for Macaulay 2*. Available at: <http://people.math.gatech.edu/~aleykin3/Dmodules>.

- [53] G. Lyubeznik, *On the local cohomology modules  $H_{\mathfrak{A}}^i(R)$  for ideals  $\mathfrak{A}$  generated by monomials in an  $R$ -sequence*, in: S. Greco and R. Strano, eds., *Complete Intersections*, Lecture Notes in Mathematics **1092**, Springer-Verlag, 1984, 214–220.
- [54] G. Lyubeznik, *A new explicit finite free resolution of ideals generated by monomials in an  $R$ -sequence*, *J. Pure Appl. Algebra* **51** (1988), 193–195.
- [55] G. Lyubeznik, *The minimal non-Cohen-Macaulay monomial ideals*, *J. Pure Appl. Algebra* **51** (1988), 261–266.
- [56] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of  $D$ -modules to commutative algebra)*, *Invent. Math.*, **113** (1993) 41–55.
- [57] G. Lyubeznik,  *$F$ -modules: applications to local cohomology and  $D$ -modules in characteristic  $p > 0$* , *J. Reine Angew. Math.* **491** (1997), 65–130.
- [58] G. Lyubeznik, *A partial survey of local cohomology modules*, en: *Local cohomology modules and its applications* (Guanajuato, 1999), 121–154, *Lecture Notes in Pure and Appl. Math.*, **226**, Dekker, New York, 2002.
- [59] G. Lyubeznik, *On the vanishing of local cohomology in characteristic  $p > 0$* , *Compositio Mat.* **142** (2006), 207–221.
- [60] G. Lyubeznik, *On some local cohomology modules*, *Adv. Math.* **213** (2007), 621–643.
- [61] G. Lyubeznik, *On some local cohomology invariants of local rings*, *Math. Z.* **254** (2006), 627–640.
- [62] B. Malgrange, *L’involutivité des caractéristiques des systèmes différentiels et microdifférentiels*, *Lect. Notes in Math.* **710**, Springer-Verlag, 1979, 277–289.
- [63] Z. Mebkhout, *Le formalisme des six opérations de Grothendieck pour les  $\mathcal{D}_X$ -modules cohérents*, *Travaux en Cours* **35**, Hermann, 1989.
- [64] E. Miller, *The Alexander Duality Functors and Local Duality with Monomial Support*, *J. Algebra* **231** (2000), 180–234.
- [65] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, *Graduate Texts in Mathematics* **227**, Springer-Verlag, New York, 2005.
- [66] M. Mustață, *Local Cohomology at Monomial Ideals*, *J. Symbolic Comput.* **29** (2000), 709–720.
- [67] M. Mustață, *Vanishing theorems on toric varieties*, *Tohoku Math. J.* **54** (2002), 451–470.
- [68] T. Oaku, *Algorithms for  $b$ -functions, restrictions and local cohomology groups of  $D$ -modules*, *Adv. in Appl. Math* **19** (1997), 61–105.
- [69] T. Oaku and N. Takayama, *Algorithms for  $D$ -modules—restriction, tensor product, localization, and local cohomology groups*, *J. Pure Appl. Algebra* **156** (2001), 267–308.
- [70] I. Peeva and M. Velasco, *Frames and degenerations of monomial resolutions*, *Trans. Amer. Math. Soc.* **363** (2011), 2029–2046.
- [71] F. Pham, *Singularités des systèmes différentiels de Gauss-Manin*, *Progress in Mathematics* **2**, Birkhäuser, 1979.
- [72] V. Reiner and V. Welker, *Linear syzygies of Stanley-Reisner ideals*, *Math. Scand.* **89** (2001), 117–132.
- [73] V. Reiner, V. Welker and K. Yanagawa, *Local cohomology of Stanley-Reisner rings with supports in general monomial ideals*, *J. Algebra* **244** (2001), 706–736.
- [74] T. Römer, *Generalized Alexander duality and applications*, *Osaka J. Math.* **38** (2001), 469–485.
- [75] T. Römer, *On minimal graded free resolutions*, PhD. Thesis, Essen (2001).
- [76] M. Sato, T. Kawai and M. Kashiwara, *Microfunctions and pseudo-differential equations*, *Lect. Notes in Math.* **287**, Springer-Verlag, 1973, 265–529.
- [77] P. Schenzel, *On Lyubeznik’s invariants and endomorphisms of local cohomology modules*, *J. Algebra* **344** (2011), 229–245.
- [78] A. M. Simon, *Some homological properties of complete modules*, *Math. Proc. Cambridge Philos. Soc.* **108** (1990), 231–246.

- [79] G. G. Smith, *Irreducible components of characteristic varieties*, J. Pure Appl. Algebra **165** (2001), 291–306.
- [80] R. P. Stanley, *Combinatorics and Commutative Algebra, Second Edition*, Progress in Mathematics **41**, Birkhäuser, 1996.
- [81] N. Terai, *Local cohomology modules with respect to monomial ideals*, Preprint 1999
- [82] D. Taylor, *Ideals generated by monomials in an  $R$ -sequence*, Ph.D. Thesis, Univ. of Chicago, 1961.
- [83] M. Velasco, *Minimal free resolutions that are not supported by a CW-complex* J. Algebra **319** (2008), 102–114.
- [84] U. Walther, *Algorithmic computation of local cohomology modules and the cohomological dimension of algebraic varieties*, J. Pure Appl. Algebra **139** (1999), 303–321.
- [85] U. Walther, *On the Lyubeznik numbers of a local ring*, Proc. Amer. Math. Soc., **129** (2001), 1631–1634
- [86] K. Yanagawa, *Alexander duality for Stanley-Reisner rings and squarefree  $\mathbb{N}^n$ -graded modules*, J. Algebra **225** (2000), 630–645.
- [87] K. Yanagawa, *Bass numbers of local cohomology modules with supports in monomial ideals*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 45–60.
- [88] K. Yanagawa, *Sheaves on finite posets and modules over normal semigroup rings*, J. Pure Appl. Algebra **161** (2001), 341–366.
- [89] K. Yanagawa, *Derived category of squarefree modules and local cohomology with monomial ideal support*, J. Math. Soc. Japan **56** (2004), 289–308.

DEPT. MATEMÀTICA APLICADA I, UNIV. POLITÈCNICA DE CATALUNYA, AV. DIAGONAL 647,  
BARCELONA 08028, SPAIN

*E-mail address:* Josep.Alvarez@upc.edu